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THE PROBLEM OF  
THE CANTILEVER PLATE

by

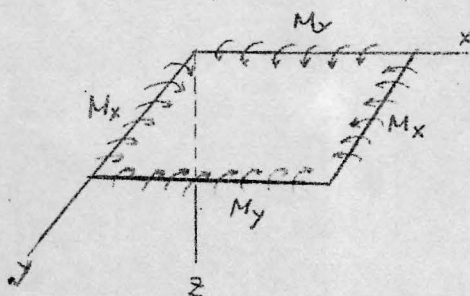
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# NOTATIONS



$x, y, z$

Rectangular coordinates

$$D = \frac{Et^3}{12(1-\sigma^2)}$$

Flexural rigidity of plates.

$M_x, M_y, M_{xy}$

Resultant bending and twisting moments per unit length in plates. The bending moments are considered positive when they make the upper side of an element of the plate compression. The twisting moment is positive when their vector sense coincide with the outer normal of the section.

$Q_x, Q_y$  Resultant shearing forces per unit length in a plate, normal to middle surface. Their sense is the same as the corresponding sense of shearing stresses.

$u, v, w$  Elastic displacements in  $x, y, z$  directions respectively.

$\psi$  Sweep-back angle.

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$\lambda, \mu$  Elastic constants.

$\sigma, E$  Poisson's ratio and Young's modulus respectively.

$$\nu = \frac{\lambda}{\mu} = \frac{2\sigma}{1-2\sigma} ; \quad A = \frac{2(\nu+1)}{\nu+2} = \frac{1}{1-\sigma}$$

$q$  Intensity of distributed load.

$n, t$  Used in subscripts denoting outer normal and tangential directions respectively.

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# THEORY OF SWEEP PLATES

## I.

### DERIVATION OF FUNDAMENTAL EQUATIONS OF DEFORMATION

#### 1.1 Introduction.

The classical theory of thin elastic plates, as formulated by Poisson (Ref. 1), Kirchhoff (Ref. 2), and Love (Refs. 3, 4, 5), is based on the following two assumptions:

(1) The state of strain is such that, any arbitrary deformation having been given to the middle surface, a line of particles originally normal to this surface remains straight and normal to it in the strained condition.

(2) The stress component normal to the middle surface of the plate is small in comparison to other stress components and may be neglected in the stress-strain relations.

These assumptions are convenient in making the mathematical formulation of the plate problem. Yet the feeling is widespread that they may introduce arbitrary constraints which are unjustifiable. Attempts have been made to give the theory a new foundation closer to the actual conditions which would permit of a systematic derivation of the fundamental differential equations governing the deformation of the plate without arbitrary assumptions or other ambiguities. In the new theory of Prof. Epstein (Ref. 6) the fundamental equations are developed in terms of the components of displacement; and<sup>in</sup> that of Synge and Chien (Refs. 7, 8) they are developed in terms of strain and curvature tensors. The latter method proves to be comprehensive and in the hands of its authors it presents a general theory of plates and shells. For small-deflection plate problems, however, it is simpler to adopt the usual scheme of development in terms of the components of displacement.

Prof. Epstein's method consists in starting from the conditions in a thick plate, regarded as a three dimensional continuum, and in carrying out the transition to the limit of a thin plate, while retaining during this process the validity of the three dimensional field equations and of the boundary conditions at the faces of the plate. We shall follow this line in the following development. The fundamental differential equations is generalized to include plates of variable thickness. This is devoted to extend the theory to actual airplane wing structures.

In explicit terms our fundamental assumptions are (1) that the thickness of the plate is small compared to other dimensions of the plate, (2) that the deformation of the plate is small compared to the thickness of the plate, and (3) that the stress tensor is analytic in  $z$ , (the coordinate normal to the middle surface of the plate),



and the expanded Taylor series converges rapidly throughout the thickness of the plate, so that good approximations are obtained by taking the first few terms.

These assumptions are, of course, tacitly implied in the classical theory of thin plates. It will readily be shown that the classical Poisson-Love theory is equivalent to the first (second order) approximation in the present theory; and that the following development provides a means to derive the fundamental equations up to any order of approximation that one may want.

## 1.2 Method of Derivation.

Let us describe the space within a thick plate by a system of orthogonal curvilinear coordinates  $x_i$ . Throughout this section the Latin indices have the range 1, 2, 3. Let the middle surface of the plate, which is a plane in the unstrained state, correspond to  $x_1 = \text{constant} = a$ , say. Let the two faces of the plate correspond to  $x_1 = a \pm \epsilon$ , resp. Denote the elastic displacement vector by  $s_i$  (or  $\bar{s}$ ), the strain tensor by  $e_{ij}$  and the stress tensor by  $p_{ij}$ . The fundamental field equation of equilibrium in an elastic continuum, when the body force is absent, consists in the vanishing of the vector divergence of the stress tensor:

$$\text{vac div } p_{ij} = 0 \quad (1)$$

Now, the stress tensor is connected to the strain tensor by the generalized Hooke's law

$$p_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} \text{ div } \bar{s} \quad (2)$$

where  $\mu$  and  $\lambda$  are the elastic constants of the material.\* Substituting (2) into (1) gives the general field equations of equilibrium in terms of the elastic displacement  $\bar{s}$ . In tensor form it is (see Ref. 9, p. 134)

$$\text{curl}(\text{curl } \bar{s}) - (\gamma + 2) \text{grad}(\text{div } \bar{s}) = 0, \quad (3)$$

where

$$\gamma = \frac{\lambda}{\mu} = \frac{2\sigma}{1-2\sigma}.$$

In this paper we are concerned with the deformation of plates under distributed loads normal to the faces of the plate. Let the loading per unit area on the faces  $x_1 = a \pm \epsilon$  be  $\pm p(x_2, x_3)$  respectively. The boundary conditions on the faces of the plate are therefore

---

\*  $\mu$  and  $\lambda$  are related to Young's modulus  $E$  and Poisson's ratio  $\sigma$  by the following relations: (see Love, Ref. 9, p. 103)

$$\left\{ \begin{array}{l} E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \\ \sigma = \frac{\lambda}{2(\lambda + \mu)} \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \mu = \frac{E}{2(1 + \sigma)} \\ \lambda = \frac{\sigma E}{(1 + \sigma)(1 - 2\sigma)} \end{array} \right.$$

$$p_{11}(a \pm \epsilon) = \pm p,$$

$$p_{12}(a \pm \epsilon) = p_{13}(a \pm \epsilon) = 0 \quad (4)$$

Equations (3) and (4), together with suitable boundary conditions along the edge of the plate, define completely the elastic deformation of a thick plate. In passing to the limit of a thin plate, we assume that, for small values of  $\epsilon$ , the stress components in eq. (4) can be expanded into Taylor's series

$$p_{11}(a \pm \epsilon) = p_{11}(a) \pm p'_{11}(a)\epsilon + p''_{11}(a)\frac{\epsilon^2}{2!} \pm \dots$$

etc., where we indicate by an accent a differentiation with respect to  $x_1$ ; and the "(a)" denotes that the attached quantity is to be evaluated at  $x_1 = a$ . Adding and subtracting the equations (4) in the expanded form, we have

$$\begin{aligned} p_{11}(a) + \frac{1}{2}\epsilon^2 p''_{11}(a) + \dots &= 0 \\ p'_{11}(a) + \frac{1}{6}\epsilon^2 p'''_{11}(a) + \dots &= \frac{p}{\epsilon} \\ p_{12}(a) + \frac{1}{2}\epsilon^2 p''_{12}(a) + \dots &= 0 \\ p'_{12}(a) + \frac{1}{6}\epsilon^2 p'''_{12}(a) + \dots &= 0 \end{aligned} \quad (5)$$

and two more eqs. obtained by changing the subscript 2 into 3 in the last two equations above.

Now the three eqs. (3) are of the second order in the components of displacement  $S_i$ . They can be resolved with respect to the three partials  $S''_1, S''_2, S''_3$  and we obtain expressions for  $S''_i$  involving only partials of zero and first order with respect to  $x_1$ . Repeated differentiation of these expressions with respect to  $x_1$  gives higher order partial derivatives of  $S_i$  with respect to  $x_1$ , all of them can be reduced to forms involving only  $S_i$  and  $S'_i$ . Now, expressing equations (5) in terms of partials of  $S_i$  and substituting the above found  $S''_i, S'''_i$  etc. ( $i = 1, 2, 3$ ), into these eqs. we obtain a system of 6 equations involving  $S'_i$ . An elimination of these 3 partials gives us a system of 3 differential equations involving only partials with respect to  $x_2$  and  $x_3$ . These are the equations for which we are searching.

Thus we obtain the fundamental equations of deformation of an elastic plate in the form of a power series of thickness of the plate. Approximations are obtained by taking the first few terms of this series.

In the following sections we shall derive the equations in Cartesian coordinates. Then they are transformed into oblique coordinates (for the convenience of dealing with swept plates). An example in bipolar coordinates, which may be useful in treating airfoils of lenticular sections, is also worked out, (Appendix A).



### 1.3 FUNDAMENTAL EQUATIONS IN CARTESIAN COORDINATES

Let  $(x, y, z)$  be a system of Cartesian coordinates, and let the plane  $z = 0$  coincide with the middle surface of the plate in unstrained state. The plate is assumed to be thin, that is, the thickness  $2\epsilon h$  of the plate is small compared to other dimensions of the plate. The thickness may be variable. We denote it by  $2\epsilon h(x, y)$  where  $2\epsilon$  is a small quantity of the dimension of the thickness and  $h(x, y)$  a function of  $x$  and  $y$  describing the thickness distribution of the plate.

The stress components in the plate are subjected to the general field equations

$$\begin{aligned} \frac{\partial}{\partial z} p_{zz} + \frac{\partial}{\partial x} p_{zx} + \frac{\partial}{\partial y} p_{zy} &= 0, \\ \frac{\partial}{\partial x} p_{xx} + \frac{\partial}{\partial y} p_{xy} + \frac{\partial}{\partial z} p_{xz} &= 0, \\ \frac{\partial}{\partial y} p_{yy} + \frac{\partial}{\partial z} p_{yz} + \frac{\partial}{\partial x} p_{yx} &= 0. \end{aligned} \quad (1)$$

$(i, j = x, y, z)$   
 $\tau_{ij,j} + F_i = 0$

These stress components are connected to the components  $(u, v, w)$  of the displacement vector  $\bar{S}$  by the relations

$$\begin{aligned} p_{zz} &= 2\mu \frac{\partial w}{\partial z} + \lambda \left( \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \\ p_{zx} &= \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \end{aligned} \quad (2)$$

and so on, by cyclic substitutions.

The field equations can be expressed in terms of the components of displacements as follows:

$$\begin{aligned} \frac{\partial^2 w}{\partial z^2} + \nabla^2 w + (1+\nu) \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0, \\ \frac{\partial^2 u}{\partial z^2} + \nabla^2 u + (1+\nu) \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0, \\ \frac{\partial^2 v}{\partial z^2} + \nabla^2 v + (1+\nu) \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0, \end{aligned} \quad (3)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The boundary conditions for a plate under distributed traction  $p(x, y)$  over its faces applied in the direction normal to the middle plane of the plate are

$$\begin{aligned} p_{zz}(\pm \epsilon h) &= \pm p, \\ p_{zx}(\pm \epsilon h) &= p_{zy}(\pm \epsilon h) = 0, \end{aligned} \quad (4)$$

where " $(\pm \epsilon h)$ " means that the quantities attached are evaluated at  $z = \pm \epsilon h$  respectively. (A total traction of  $2p$  per unit area of the middle

surface is distributed half on top face and half on bottom face of the plate). (See footnote below).

Footnote. Since the middle surface of a plate under the boundary conditions  $p_{zz}(\pm \epsilon h) = p$  is obviously undeflected, i.e.  $w(0) = 0$ , we infer by the principle of superposition that the differential equations derived from the boundary conditions (4) is the same as that derived from that derived from the conditions

$$p_{zz}(\pm \epsilon h) = 2p,$$

$$p_{zz}(-\epsilon h) = 0.$$

We may demand, however, that the boundary conditions are such that the tractions  $\pm p(x, y)$  are applied over the faces of the plate in the direction of the normal to the faces, as in the case of air pressure over a wing. The conditions are, accordingly,

$$p_{zz}(\pm \epsilon h) = \pm p,$$

$$p_{zx}(\pm \epsilon h) = \epsilon p \frac{\partial h}{\partial x},$$

$$p_{zy}(\pm \epsilon h) = \epsilon p \frac{\partial h}{\partial y}.$$

Or, in expanded form correspond to eqs. (5),

$$p_{zx}(0) + \frac{\epsilon^2 h^2}{2!} p_{zx}''(0) + \dots = 2 \epsilon p \frac{\partial h}{\partial x},$$

$$p_{zy}(0) + \frac{\epsilon^2 h^2}{2!} p_{zy}''(0) + \dots = 2 \epsilon p \frac{\partial h}{\partial y}.$$

All the rest of eqs. (5) are unchanged.

The first eq. of (9) is changed to have on its left hand side a quantity

$$\frac{p}{\epsilon h} + \epsilon \frac{\partial}{\partial x} \left( p \frac{\partial h}{\partial x} \right) + \epsilon \frac{\partial}{\partial y} \left( p \frac{\partial h}{\partial y} \right) \quad (a)$$

instead of the original  $\frac{p}{\epsilon h}$ .

It may be observed that the last two equations of (11) and hence (18) are unaffected by this change of boundary conditions. Therefore we conclude that the final result for the above named case can be obtained by changing  $p/\epsilon h$  on the left hand side of the first eq. of (25) into the quantity (a) above.



Assuming that the stress components are analytic throughout the thickness, we obtain, by adding and subtracting the expanded form of (4) the equations

$$\begin{aligned}
 p_{zz}(0) + \frac{1}{2!} \epsilon^2 h^2 p_{zz}''(0) + \frac{1}{4!} \epsilon^4 h^4 p_{zz}^{(IV)}(0) + \dots &= 0, \\
 p_{zz}'(0) + \frac{1}{3!} \epsilon^2 h^2 p_{zz}'''(0) + \frac{1}{5!} \epsilon^4 h^4 p_{zz}^{(V)}(0) + \dots &= \frac{p}{\epsilon h}, \\
 p_{zx}(0) + \frac{1}{2!} \epsilon^2 h^2 p_{zx}''(0) + \frac{1}{4!} \epsilon^4 h^4 p_{zx}^{(IV)}(0) + \dots &= 0, \\
 p_{zx}'(0) + \frac{1}{3!} \epsilon^2 h^2 p_{zx}'''(0) + \frac{1}{5!} \epsilon^4 h^4 p_{zx}^{(V)}(0) + \dots &= 0
 \end{aligned} \tag{5}$$

and two more eqs. obtained from the last two by changing the subscript  $x$  into  $y$ . The accents denote differentiation with respect to  $z$ .

Now the stress component

$$p_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{2a}$$

is defined exclusively by partials with respect to  $x$  and  $y$ . But  $p_{xx}$  and  $p_{yy}$  contain partial derivative with respect to  $z$ . Let us denote by

$$A \equiv \frac{2(\gamma+1)}{\gamma+2} = \frac{1}{1-\sigma} \tag{6}$$

and define the two-dimensional stress components

$$\begin{aligned}
 q_{xx} &\equiv 2\mu \left[ A \frac{\partial u}{\partial x} + (A-1) \frac{\partial v}{\partial y} \right], \\
 q_{yy} &\equiv 2\mu \left[ (A-1) \frac{\partial u}{\partial x} + A \frac{\partial v}{\partial y} \right],
 \end{aligned} \tag{7}$$

so that

$$\begin{aligned}
 p_{xx} &\equiv (A-1) p_{zz} + q_{xx}, \\
 p_{yy} &\equiv (A-1) p_{zz} + q_{yy}.
 \end{aligned} \tag{8}$$

Substituting (5) and (8) into the field eqs. (1) which are certainly valid for the particular value of  $z = 0$ , we obtain, up to fourth order in  $\epsilon$ ,

$$\frac{p}{\epsilon h} = \frac{\epsilon^2}{2!} \left\{ \frac{h^2}{3} p_{zz}''' + \frac{\partial}{\partial x} [h^2 p_{zx}'] + \frac{\partial}{\partial y} [h^2 p_{zy}'] \right\} +$$

$$+ \frac{\epsilon^4}{4!} \left\{ \frac{h^4}{5} p_{zz}^{(IV)} + \frac{\partial}{\partial x} [h^4 p_{zx}^{(IV)}] + \frac{\partial}{\partial y} [h^4 p_{zy}^{(IV)}] \right\},$$

$$\frac{\partial}{\partial x} q_{xx} + \frac{\partial}{\partial y} p_{xy} - \frac{\epsilon^2}{2!} \left\{ (A-1) \frac{\partial}{\partial x} (h^2 p_{zz}') + \frac{1}{3} h^2 p_{zx}''' \right\} -$$

$$- \frac{\epsilon^4}{4!} \left\{ (A-1) \frac{\partial}{\partial x} (h^4 p_{zz}^{(IV)}) + \frac{1}{5} h^4 p_{zx}^{(IV)} \right\} = 0,$$

$$\frac{\partial}{\partial y} q_{yy} + \frac{\partial}{\partial x} p_{xy} - \frac{\epsilon^2}{2!} \left\{ (A-1) \frac{\partial}{\partial y} (h^2 p_{zz}') + \frac{1}{3} h^2 p_{zy}''' \right\}$$

$$- \frac{\epsilon^4}{4!} \left\{ (A-1) \frac{\partial}{\partial y} (h^4 p_{zz}^{(IV)}) + \frac{1}{5} h^4 p_{zy}^{(IV)} \right\} = 0$$

(9)

where all the partial derivatives are evaluated at  $z = 0$ .

If we express all the partial derivatives of the stress components in this eq. in terms of partials with respect to  $x$  and  $y$  of the displacement components, we shall obtain the fundamental differential equations defining the deformation of the elastic plate. We shall calculate these eqs. up to fourth order in  $\epsilon$ . This requires that  $p_{zz}'$ ,  $p_{zx}'$ ,  $p_{zy}'$ ,  $q_{zz}'$ ,  $p_{zx}''$ ,  $p_{zy}''$  be accurate up to second order in  $\epsilon$ , and  $p_{zz}^{(IV)}$ ,  $p_{zx}^{(IV)}$ , etc. up to zero order. Now according to the definitions (2) and conditions (5) we have, when  $z = 0$ :

$$\frac{\partial w}{\partial z} = -(A-1) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{(2-A)}{4\mu} \epsilon^2 h^2 p_{zz}' ,$$

$$\frac{\partial u}{\partial z} = - \frac{\partial w}{\partial x} - \frac{1}{2!} \frac{\epsilon^2 h^2}{\mu} p_{zx}'' ,$$

$$\frac{\partial v}{\partial z} = - \frac{\partial w}{\partial y} - \frac{1}{2!} \frac{\epsilon^2 h^2}{\mu} p_{zy}'' .$$

(10)

From the field eqs. (3), and using (10), we have

$$\frac{\partial^2 w}{\partial z^2} = (A-1) \nabla^2 w + \frac{A\epsilon^2}{4\mu} \left\{ \frac{\partial}{\partial x} (h^2 p_{zx}') + \frac{\partial}{\partial y} (h^2 p_{zy}') \right\},$$

$$\frac{\partial^2 u}{\partial z^2} = - \nabla^2 u - A \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{A}{4\mu} \epsilon^2 \frac{\partial}{\partial x} (h^2 p_{zz}'),$$

$$\frac{\partial^2 v}{\partial z^2} = - \nabla^2 v - A \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{A}{4\mu} \epsilon^2 \frac{\partial}{\partial y} (h^2 p_{zz}').$$

(11)



But from (1) and (8), the field eqs. can be written as

$$\begin{aligned} p'_{zz} + \frac{\partial}{\partial x} p_{zx} + \frac{\partial}{\partial y} p_{zy} &= 0, \\ p'_{xz} + \frac{\partial}{\partial x} q_{xx} + (A-1) \frac{\partial}{\partial x} p_{zz} + \frac{\partial}{\partial y} p_{xy} &= 0, \\ p'_{yz} + \frac{\partial}{\partial y} q_{yy} + (A-1) \frac{\partial}{\partial y} p_{zz} + \frac{\partial}{\partial x} p_{xy} &= 0. \end{aligned} \quad (12)$$

The partials  $p_{zz}''$ ,  $p_{zz}'''$ , etc. can be found by successive differentiation of this eq. with respect to  $z$ . Note that we need  $p_{zz}''$  etc. correct only to zero order in (10) and (11), we shall evaluate them to zero order first.

From (2a), (7) and (10) we have

$$\begin{aligned} p'_{xy} &= -2\mu \frac{\partial^2 w}{\partial x \partial y}, \\ q'_{xx} &= -2\mu \left\{ A \nabla^2 w - \frac{\partial^2 w}{\partial y^2} \right\}, \\ q'_{yy} &= -2\mu \left\{ A \nabla^2 w - \frac{\partial^2 w}{\partial x^2} \right\}. \end{aligned} \quad (13)$$

Hence by differentiating (12) with respect to  $z$ , and using (13), (2) and (7), we obtain:

$$\begin{aligned} \frac{1}{2\mu} p''_{zz} &= A \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \\ \frac{1}{2\mu} p''_{zx} &= A \nabla^2 \frac{\partial w}{\partial x}, \\ \frac{1}{2\mu} p''_{zy} &= A \nabla^2 \frac{\partial w}{\partial y}. \end{aligned} \quad (14)$$

From which we can write (10) and (11) in the following form:

$$\begin{aligned} \frac{\partial w}{\partial z} &= -(A-1) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{A(2-A)}{2} \epsilon^2 h^2 \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \\ \frac{\partial u}{\partial z} &= - \frac{\partial w}{\partial x} - A \epsilon^2 h^2 \nabla^2 \frac{\partial w}{\partial x}, \\ \frac{\partial v}{\partial z} &= - \frac{\partial w}{\partial y} - A \epsilon^2 h^2 \nabla^2 \frac{\partial w}{\partial y}. \end{aligned} \quad (15)$$

$$\begin{aligned}
\frac{\partial^2 w}{\partial z^2} &= (A-1) \nabla^2 w + \frac{A^2 \epsilon^2}{2} \left\{ \frac{\partial}{\partial x} (h^2 \nabla^2 \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y} (h^2 \nabla^2 \frac{\partial w}{\partial y}) \right\}, \\
\frac{\partial^2 u}{\partial z^2} &= -\nabla^2 u - A \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{A^2 \epsilon^2}{2} \frac{\partial}{\partial x} \left[ h^2 \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right], \\
\frac{\partial^2 v}{\partial z^2} &= -\nabla^2 v - A \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{A^2 \epsilon^2}{2} \frac{\partial}{\partial y} \left[ h^2 \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right].
\end{aligned} \tag{16}$$

Substituting (15) into the results of differentiation of the definitions (2a) and (7) we obtain, similar to (13) but now correct to second order in  $\epsilon$  :

$$\begin{aligned}
\frac{1}{2\mu} p'_{xy} &= -\frac{\partial^2 w}{\partial x \partial y} - \frac{A \epsilon^2}{2} \left\{ \frac{\partial}{\partial y} (h^2 \nabla^2 \frac{\partial w}{\partial x}) + \frac{\partial}{\partial x} (h^2 \nabla^2 \frac{\partial w}{\partial y}) \right\}, \\
\frac{1}{2\mu} \phi'_{xx} &= -A \nabla^2 w + \frac{\partial^2 w}{\partial y^2} - A \epsilon^2 \left\{ A \frac{\partial}{\partial x} (h^2 \nabla^2 \frac{\partial w}{\partial x}) + (A-1) \frac{\partial}{\partial y} (h^2 \nabla^2 \frac{\partial w}{\partial y}) \right\}, \\
\frac{1}{2\mu} q'_{yy} &= -A \nabla^2 w + \frac{\partial^2 w}{\partial x^2} - A \epsilon^2 \left\{ A \frac{\partial}{\partial y} (h^2 \nabla^2 \frac{\partial w}{\partial y}) + (A-1) \frac{\partial}{\partial x} (h^2 \nabla^2 \frac{\partial w}{\partial x}) \right\}.
\end{aligned} \tag{17}$$

Similarly by a second differentiation of (2a) and (7) with respect to  $z$  and using (16), we obtain

$$\begin{aligned}
\frac{1}{2\mu} p''_{xy} &= -\frac{1}{2} \nabla^2 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - A \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{A^2 \epsilon^2}{2} \frac{\partial^2}{\partial x \partial y} \left[ h^2 \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right], \\
\frac{1}{2\mu} q''_{xx} &= -A(1+A) \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \nabla^2 \frac{\partial v}{\partial y} + A \frac{\partial^2}{\partial y^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\
&\quad + \frac{A^2 \epsilon^2}{2} \left( A \nabla^2 - \frac{\partial^2}{\partial y^2} \right) \left[ h^2 \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right], \\
\frac{1}{2\mu} q''_{yy} &= -A(1+A) \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \nabla^2 \frac{\partial u}{\partial x} + A \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\
&\quad + \frac{A^2 \epsilon^2}{2} \left( A \nabla^2 - \frac{\partial^2}{\partial x^2} \right) \left[ h^2 \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right].
\end{aligned} \tag{18}$$



Differentiate once and twice the field eqs. (3) with respect to  $z$ , and using (15) and (16), we obtain, correct to zero order in  $\epsilon$ ,

$$\frac{\partial^3 w}{\partial z^3} = (2A-1) \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$\frac{\partial^3 u}{\partial z^3} = (1+A) \nabla^2 \frac{\partial w}{\partial x},$$

$$\frac{\partial^3 v}{\partial z^3} = (1+A) \nabla^2 \frac{\partial w}{\partial y};$$

$$\frac{\partial^4 w}{\partial z^4} = (1-2A) \nabla^4 w,$$

$$\frac{\partial^4 u}{\partial z^4} = \nabla^4 u + 2A \nabla^2 \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$\frac{\partial^4 v}{\partial z^4} = \nabla^4 v + 2A \nabla^2 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right).$$

(19)

From these we obtain, by a third and fourth partial differentiation of (2a) and (7) with respect to  $z$ , the following expressions correct to zero order in  $\epsilon$ :

$$\frac{1}{2\mu} p_{xy}''' = (1+A) \nabla^2 \frac{\partial^2 w}{\partial x \partial y},$$

$$\frac{1}{2\mu} q_{xx}''' = A(1+A) \nabla^4 w - (1+A) \nabla^2 \frac{\partial^2 w}{\partial y^2},$$

$$\frac{1}{2\mu} q_{yy}''' = A(1+A) \nabla^4 w - (1+A) \nabla^2 \frac{\partial^2 w}{\partial x^2},$$

$$\frac{1}{2\mu} p_{xy}^{(IV)} = \frac{1}{2} \nabla^4 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + 2A \nabla^2 \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$\frac{1}{2\mu} q_{xx}^{(IV)} = A(2A+1) \nabla^4 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \nabla^4 \frac{\partial v}{\partial y} - 2A \nabla^2 \frac{\partial^2}{\partial y^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$\frac{1}{2\mu} q_{yy}^{(IV)} = A(2A+1) \nabla^4 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \nabla^4 \frac{\partial u}{\partial x} - 2A \nabla^2 \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right).$$

(20)

Now we are in a position to calculate  $p_{zz}''$ ,  $p_{zz}'''$ , etc. From (12) and using (2a) and (7) and (15) we obtain, correct to second order in  $\epsilon$  :

$$\begin{aligned}\frac{1}{2\mu} p_{zz}' &= \frac{A\epsilon^2}{2} \left\{ \frac{\partial}{\partial x} (h^2 \nabla^2 \frac{\partial W}{\partial x}) + \frac{\partial}{\partial y} (h^2 \nabla^2 \frac{\partial W}{\partial y}) \right\}, \\ \frac{1}{2\mu} p_{zx}' &= -A \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial^2 u}{\partial y^2} + (\frac{1}{2} - A) \frac{\partial^2 v}{\partial x \partial y} + \frac{A(A-1)}{2} \epsilon^2 \frac{\partial}{\partial x} [h^2 \nabla^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})], \\ \frac{1}{2\mu} p_{zy}' &= -A \frac{\partial^2 v}{\partial y^2} - \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + (\frac{1}{2} - A) \frac{\partial^2 u}{\partial x \partial y} + \frac{A(A-1)}{2} \epsilon^2 \frac{\partial}{\partial y} [h^2 \nabla^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})].\end{aligned}\quad (21)$$

Differentiating (12) with respect to  $z$ , and using (17) and (21), we obtain, correct to second order in  $\epsilon$ ,

$$\begin{aligned}\frac{1}{2\mu} p_{zz}'' &= A \nabla^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \frac{A(A-1)}{2} \epsilon^2 \nabla^2 [h^2 \nabla^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})], \\ \frac{1}{2\mu} p_{zx}'' &= A \nabla^2 \frac{\partial W}{\partial x} + \frac{A\epsilon^2}{2} \left\{ (A \frac{\partial^2}{\partial x^2} + \nabla^2) (h^2 \nabla^2 \frac{\partial W}{\partial x}) + A \frac{\partial^2}{\partial x \partial y} (h^2 \nabla^2 \frac{\partial W}{\partial y}) \right\}, \\ \frac{1}{2\mu} p_{zy}'' &= A \nabla^2 \frac{\partial W}{\partial y} + \frac{A\epsilon^2}{2} \left\{ (A \frac{\partial^2}{\partial y^2} + \nabla^2) (h^2 \nabla^2 \frac{\partial W}{\partial y}) + A \frac{\partial^2}{\partial x \partial y} (h^2 \nabla^2 \frac{\partial W}{\partial x}) \right\}.\end{aligned}\quad (22)$$

Further differentiation of (12) leads to

$$\begin{aligned}\frac{1}{2\mu} p_{zz}''' &= -A \nabla^4 W - \frac{A\epsilon^2}{2} \left\{ \frac{\partial}{\partial x} [(A \frac{\partial^2}{\partial x^2} + \nabla^2) (h^2 \nabla^2 \frac{\partial W}{\partial x}) + A \frac{\partial^2}{\partial x \partial y} (h^2 \nabla^2 \frac{\partial W}{\partial y})] \right. \\ &\quad \left. + \frac{\partial}{\partial y} [(A \frac{\partial^2}{\partial y^2} + \nabla^2) (h^2 \nabla^2 \frac{\partial W}{\partial y}) + A \frac{\partial^2}{\partial x \partial y} (h^2 \nabla^2 \frac{\partial W}{\partial x})] \right\}, \\ \frac{1}{2\mu} p_{zx}''' &= 2A \frac{\partial}{\partial x} \nabla^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \frac{1}{2} \nabla^2 \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} \nabla^2 \frac{\partial^2 u}{\partial y^2} \\ &\quad - \frac{A\epsilon^2}{2} (2A-1) \frac{\partial}{\partial x} \nabla^2 [h^2 \nabla^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})],\end{aligned}\quad (23)$$

and  $\frac{1}{2\mu} p_{zy}'''$  obtained by interchanging  $u, v$ , and  $x, y$  in the expression for  $\frac{1}{2\mu} p_{zx}'''$  above,



and the following expressions in zero order:

$$\frac{1}{2\mu} p_{zz}^{(IV)} = -2A \nabla^4 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$\frac{1}{2\mu} p_{zx}^{(IV)} = -2A \nabla^4 \frac{\partial w}{\partial x},$$

$$\frac{1}{2\mu} p_{zy}^{(IV)} = -2A \nabla^4 \frac{\partial w}{\partial y},$$

$$\frac{1}{2\mu} p_{zz}^{(V)} = 2A \nabla^6 w,$$

$$\frac{1}{2\mu} p_{zx}^{(V)} = -3A \nabla^4 \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{1}{2} \nabla^4 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right),$$

$$\frac{1}{2\mu} p_{zy}^{(V)} = -3A \nabla^4 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{1}{2} \nabla^4 \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).$$

(24)

Now all the quantities in eq. (9) are obtained in required form. A single step of substituting them into equation (9) gives us the final result:

$$\begin{aligned} \frac{1}{\epsilon h} = & \mu A \epsilon^2 \left\{ -\frac{h^2}{3} \nabla^4 w + \frac{\partial}{\partial x} (h^2 \nabla^2 \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y} (h^2 \nabla^2 \frac{\partial w}{\partial y}) \right\} + \\ & + \frac{2\mu \epsilon^4}{4!} \left\{ \left( 6 \frac{\partial h^2}{\partial x} + 4h^2 \frac{\partial}{\partial x} \right) \left[ \left( A \frac{\partial^2}{\partial x^2} + \nabla^2 \right) (h^2 \nabla^2 \frac{\partial w}{\partial x}) + A \frac{\partial^2}{\partial x \partial y} (h^2 \nabla^2 \frac{\partial w}{\partial y}) \right] \right. \\ & + \left( 6 \frac{\partial h^2}{\partial y} + 4h^2 \frac{\partial}{\partial y} \right) \left[ \left( A \frac{\partial^2}{\partial y^2} + \nabla^2 \right) (h^2 \nabla^2 \frac{\partial w}{\partial y}) + A \frac{\partial^2}{\partial x \partial y} (h^2 \nabla^2 \frac{\partial w}{\partial x}) \right] \\ & \left. + \frac{2h^4}{5} \nabla^6 w - 2A \frac{\partial}{\partial x} (h^4 \nabla^4 \frac{\partial w}{\partial x}) - 2A \frac{\partial}{\partial y} (h^4 \nabla^4 \frac{\partial w}{\partial y}) \right\}, \end{aligned}$$

$$\begin{aligned} & A \frac{\partial^2 u}{\partial x^2} + (A - \frac{1}{2}) \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} - \\ & - \frac{\epsilon^2}{2!} \left\{ \left[ A(A-1) \frac{\partial h^2}{\partial x} + A(A-\frac{1}{3}) h^2 \frac{\partial}{\partial x} \right] \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{6} h^2 \nabla^2 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right\} \\ & + \frac{\epsilon^4}{4!} \left\{ \left[ 6A(A-1)^2 \frac{\partial h^2}{\partial x} + 2A(3A^2 - 4A + 2) h^2 \frac{\partial}{\partial x} \right] \nabla^2 \left[ h^2 \nabla^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \right. \\ & \left. + \left[ 2A(A-1) \frac{\partial h^4}{\partial x} + \left\{ 2A(A-1) + \frac{3}{5}A \right\} h^4 \frac{\partial}{\partial x} \right] \nabla^4 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{h^4}{10} \nabla^4 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right\} \\ & = 0, \end{aligned}$$

and a third equation obtained from the second eq. of (25) by interchanging  $x$  and  $y$ , and  $u$  and  $v$ .

### Particular case of flat plate

In the particular case of a flat plate with constant thickness  $t$ , the eqs. (25) is simplified down to the following form:

$$h = 1, \quad 2\epsilon = t,$$

$$\frac{\mu A t^2}{6} \nabla^4 w + \frac{\mu t^4}{8 \lambda 4!} (2A + \frac{22}{5}) \nabla^6 w = \frac{2P}{t},$$

$$\begin{aligned} & A \frac{\partial^2 u}{\partial x^2} + (A - \frac{1}{2}) \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} - \frac{1}{2!} \left(\frac{t}{2}\right)^2 \left\{ A(A - \frac{1}{3}) \nabla^2 \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{6} \nabla^2 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right\} \\ & + \frac{1}{4!} \left(\frac{t}{2}\right)^4 \left\{ A(6A^2 - 6A + \frac{13}{5}) \nabla^4 \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{10} \nabla^4 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \right\} \\ & = 0 \end{aligned}$$

(26)

and a third equation obtained from the second equation of (26) by interchanging  $x$  and  $y$ , and  $u$  and  $v$ .

It is remarkable that the differential equation defining  $w$  and those defining  $u$ ,  $v$  are independent of each other. Therefore when we are interested only in the normal deflection of an elastic plate, we need only to solve a single equation in  $w$ .

When the fourth order terms in  $\epsilon$  are neglected, our fundamental differential eq. in  $w$  is

$$-\frac{h^2}{3} \nabla^4 w + \frac{\partial}{\partial x} (h^2 \nabla^2 \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y} (h^2 \nabla^2 \frac{\partial w}{\partial y}) = \frac{P}{\mu A h \epsilon^3}.$$

(27)



This reduces to the familiar form

$$\nabla^4 w = \frac{2p}{D}, \quad (28)$$

$$\text{where } D = Et^3/12(1-\sigma^2).$$

When the thickness  $2 \leq h$  of the plate is uniform and equal to  $t$ .

It may be remarked that the fundamental differential eq. (27) for a plate of variable thickness is linear with variable coefficients. It does not seem to be deducible from the Poisson-Love theory. Write (27) in the form

$$\nabla^4 w + 3 \left\{ \frac{\partial \log h}{\partial x} \nabla^2 \frac{\partial w}{\partial x} + \frac{\partial \log h}{\partial y} \nabla^2 \frac{\partial w}{\partial y} \right\} = \frac{2p}{D}, \quad (27a)$$

and remembering that  $-D \nabla^2 \frac{\partial w}{\partial x}$  and  $-D \nabla^2 \frac{\partial w}{\partial y}$  are equal to the vertical shear force per unit length in the plate,  $Q_x$  and  $Q_y$ , respectively, we see that the second and third term on the right hand side of (27a) represent the effect of components of shear forces in  $z$ -direction, contributing essentially to a change in  $p$ .

### Boundary Conditions

The boundary conditions at the edges of the plate should be specified in consistence with the order of approximation adopted in the differential equations of equilibrium. In a thick plate subjected to given forces either the stresses or the displacements have prescribed values at every point of the edge. For example, if one edge of a thick plate coincides with the surface  $x = \text{const.}$ , the boundary conditions at that edge are

1.  $u = v = w = 0$  for clamped edges,
2.  $p_{xx} = p_{xy} = p_{xz} = 0$  for free edges, and
3. Some mixed conditions for supported edges.

(29)

In passing the field equations to the limit of a thin plate, these boundary conditions pass to their respective integrated forms too. Thus in the first approximation, when our differential equation is of fourth order, we need two conditions at each edge. These may be specified by

1.  $w = \frac{\partial w}{\partial n} = 0$  for clamped edges,
2.  $w$  and bending moment  $= 0$  for simply supported edges,
3. Bending moment  $= 0$ , Resultant reaction  $= 0$  for free edges, where the bending moments and resultant reactions are to be calculated only to zero order in the thickness of the plate.

(30)

In the second approximation, we have a differential equation of sixth order, so we need three conditions at each edge. These may be specified, for example, by

1.  $w = 0$ ,  $\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} - A e^2 h^2 \nabla^2 \frac{\partial w}{\partial x} = 0$ ,  $\frac{\partial^3 u}{\partial z^3} = (1+A) \nabla^2 \frac{\partial w}{\partial x} = 0$   
for <sup>a</sup>clamped edge,  $x = \text{const.}$ ,
2.  $w = 0$ , bending moment = 0, twisting moment = 0, for simply supported edges,
3. Bending moment = twisting moment = vertical shear = 0, for free edges (31).

while the quantities named in (31) are now to be calculated up to second order in the thickness of the plate.

For further higher order approximations we have still higher order differential equations and we need more stringent specifications of integrated forms of boundary conditions.

Explicit expressions of forces and moments can be easily obtained as follows. Consider the edge  $x = \text{const.}$  According to our fundamental assumption, the stresses are expandable into power series in  $z$  throughout the thickness of the plate. Hence

$$p_{xx} = p_{xx}(0) + p'_{xx}(0)z + p''_{xx}(0)\frac{z^2}{2!} + \dots \quad (32)$$

$$p_{xy} = p_{xy}(0) + p'_{xy}(0)z + p''_{xy}(0)\frac{z^2}{2!} + \dots$$

etc. Now the resultant force  $\bar{X} = \int_{-eh}^{+eh} p_{xx} dz$

Bending moment  $M_x = \int_{-eh}^{+eh} p_{xx} z dz$

Vertical shear  $Q_x = \int_{-eh}^{+eh} p_{xz} dz$  (33)

Twisting moment  $M_{xy} = - \int_{-eh}^{+eh} p_{xy} z dz$

Hence

$$\begin{aligned} \bar{X} &= 2eh p_{xx}(0) + 2 \frac{(eh)^3}{3!} p''_{xx}(0) + \dots \\ M_x &= 2 \frac{(eh)^3}{3} p'_{xx}(0) + \frac{2(eh)^5}{5 \times 3!} p'''_{xx}(0) + \dots \end{aligned} \quad (34)$$

$$Q_x = 2eh p_{xz}(0) + \frac{2(eh)^3}{3!} p'''_{xz}(0) + \dots$$

$$M_{xy} = -2 \frac{(eh)^3}{3} p'_{xy}(0) - 2 \frac{(eh)^5}{5 \times 3!} p'''_{xy}(0) + \dots$$

The quantities  $p_{xx}(0)$ ,  $p_{xx}'(0)$ ,  $p_{xx}''(0)$ , etc. should be consistently evaluated according to the order of approximation used. For second approximations (up to 4th order in  $\epsilon$ ) they are given by eqs. (2), (8), (17), (18), (20), (21), (22), and (23). They reduce to the familiar forms in the first approximation as follows:

$$\begin{aligned} M_x &= \frac{2(\epsilon h)^3}{3} p_{xx}'(0) = -D \left( \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} \right), \\ Q_x &= 2\epsilon h p_{xz}(0) + \frac{2(\epsilon h)^3}{3!} p_{xz}''(0) = -D \nabla^2 \frac{\partial w}{\partial x}, \\ M_{xy} &= -\frac{2}{3} (\epsilon h)^3 p_{xy}'(0) = D (1-\sigma) \frac{\partial^2 w}{\partial x \partial y}, \end{aligned} \quad (35)$$

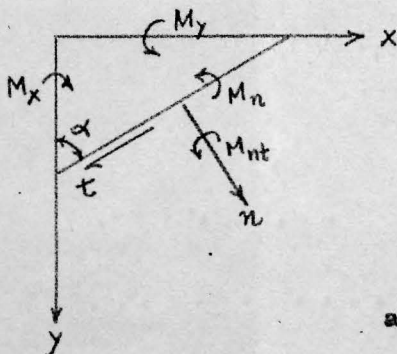
where  $D = \frac{Et^3}{12(1-\sigma^2)}$ ,  $t = 2\epsilon h$ .

The resultant reaction referred to in the third line of (30) leads to the well-known Kirchhoff's condition for free edges. (See Ref. 9, or 10, or 11 for explanation and historical notes.) In short, the statically equipollent resultant shearing force on the edge  $x = \text{const}$  is a distributed load of

$$Q_x - \frac{\partial M_{xy}}{\partial y} = -D \left\{ \frac{\partial^3 w}{\partial x^3} + (2-\sigma) \frac{\partial^3 w}{\partial x \partial y^2} \right\} \quad (36)$$

and a concentrated force of magnitude  $M_{xy}$  at the corners. The last point was first pointed out by H. Lamb. (Ref. 4) from the variational procedure of Kirchhoff and is readily implied by Thomson and Tait's explanation of Kirchhoff's condition.

Similar formulas for forces and moments on other edges are easily obtained by coordinate transformations.



If  $\alpha$  is the angle between the normal  $n$  of a section to the x-axis, or between the direction  $t$  and the y-axis, considered positive when measured in clockwise direction, then

$$\begin{aligned} M_n &= M_x \cos^2 \alpha + M_y \sin^2 \alpha, \\ M_{nt} &= \frac{1}{2} \sin 2\alpha (M_x - M_y), \end{aligned} \quad (37)$$

and in first approximation,

$$\begin{aligned} M_n &= -D \left( \frac{\partial^2 w}{\partial n^2} + \sigma \frac{\partial^2 w}{\partial t^2} \right), \\ M_{nt} &= D (1-\sigma) \frac{\partial^2 w}{\partial n \partial t}. \end{aligned} \quad (38)$$



## Calculation of Stresses

Knowing the deflection surface  $w(x,y)$  of the plate we can calculate the stresses in the plate due to bending of the plate by series expansions like eqs. (32). Of course the consistency in the order of approximation must be observed.

For the first approximation, we have the formulas for the maximum normal stresses

$$(\sigma_x)_{\max} = \frac{6 M_x}{t^2}, \quad (\sigma_y)_{\max} = \frac{6 M_y}{t^2}. \quad (39)$$

The shearing stresses parallel to x- and y-axes are obtained from

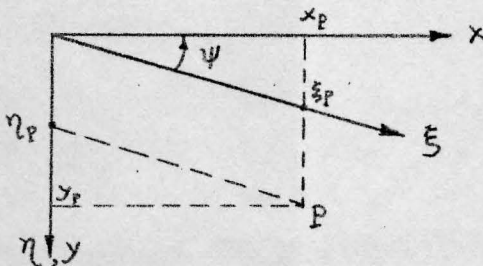
$$(\tau_{xy})_{\max} = - G t \frac{\partial^2 w}{\partial x \partial y} \quad (40)$$

and the shearing stresses parallel to the z-axis are obtained from the fact that the shearing forces  $Q_x$  and  $Q_y$  are distributed along the thickness of the plate following the parabolic law, (Cf. eq. (32) and (34), and remembering that we are talking about the first approximation.) as in the case of beams of rectangular cross section,

$$(\tau_{xz})_{\max} = \frac{3}{2h} Q_{x\max}, \quad (\tau_{yz})_{\max} = \frac{3}{2h} Q_{y\max}. \quad (41)$$

### 1.4 EQUATIONS OF DEFORMATION IN OBLIQUE COORDINATES.

Oblique coordinates may be used to advantage in treating the problems of swept plates. By oblique coordinates we mean a coordinate system in which the coordinates of a point P are given by distances measured parallel to  $\xi$  and  $\eta$  axes.



Let  $x, y$  be a rectangular coordinate system. Let  $\eta$ -axis be coincident with the  $y$  axis, and  $\xi$ -axis be at an angle  $\psi$  from

the  $x$ -axis.  $\psi$  is said to be positive when it is measured counter-clockwise from  $\xi$ - to  $x$ -axes.

It is clear from the figure that the coordinate transformation is given by the following equations

$$\begin{cases} x = \xi \cos \psi \\ y = \eta + \xi \sin \psi \end{cases}, \quad \text{or} \quad \begin{cases} \xi = x \sec \psi \\ \eta = y - x \tan \psi \end{cases}. \quad (1)$$

from which

$$\begin{aligned}
 ds^2 &= dx^2 + dy^2 = d\xi^2 + d\eta^2 + 2 \sin \psi d\xi d\eta, \\
 \frac{\partial}{\partial x} &= \frac{1}{\cos \psi} \left( \frac{\partial}{\partial \xi} - \sin \psi \frac{\partial}{\partial \eta} \right), \\
 \frac{\partial}{\partial y} &= \frac{\partial}{\partial \eta}, \\
 \frac{\partial^2}{\partial x^2} &= \frac{1}{\cos^2 \psi} \left( \frac{\partial^2}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2}{\partial \xi \partial \eta} + \sin^2 \psi \frac{\partial^2}{\partial \eta^2} \right),
 \end{aligned} \tag{2}$$

and so on.

Because of the complication in mathematics induced by an oblique angle, let us be satisfied with ourself in a general survey of the swept plate problem with the first approximation which is identical with the classical Poisson-Love theory of plates. The fundamental differential equations of deformation are given by eq. (1.3:27) for plates of variable thickness and by eq. (1.3:28) for plates of constant thickness. These equations can be transformed into oblique coordinates.

For flat plates with constant thickness  $t$ , equation (1.3:28)

$$\nabla^4 w = \frac{2p}{D}$$

becomes, in oblique coordinates,

$$\frac{\partial^4 w}{\partial \xi^4} - 4 \sin \psi \frac{\partial^4 w}{\partial \xi^3 \partial \eta} + (2 + 4 \sin^2 \psi) \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} - 4 \sin \psi \frac{\partial^4 w}{\partial \xi \partial \eta^3} + \frac{\partial^4 w}{\partial \eta^4} = k, \tag{3}$$

where  $k = \frac{2p \cos^4 \psi}{D}$ ,

$2p$  = the load per unit area of the plate

$$D = \frac{Et^3}{12(1-\sigma^2)}$$

This can also be written as

$$\left( \frac{\partial^2}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right) \left( \frac{\partial^2 w}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2} \right) = k.$$

For plates of variable thickness, the equation (1.3:27a)

$$\nabla^4 w + 3 \left\{ \frac{\partial \log h}{\partial x} \nabla^2 \frac{\partial w}{\partial x} + \frac{\partial \log h}{\partial y} \nabla^2 \frac{\partial w}{\partial y} \right\} = \frac{2p}{D}$$

where  $D = E(2\epsilon h)^3 / 12(1-\sigma^2)$  is the local flexural rigidity of the plates becomes

$$\begin{aligned}
& \left( \frac{\partial^2}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right)^2 w \\
& + 3 \left\{ \left[ \left( \frac{\partial}{\partial \xi} - \sin \psi \frac{\partial}{\partial \eta} \right) \log h \right] \left( \frac{\partial^2}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right) \left( \frac{\partial w}{\partial \xi} - \sin \psi \frac{\partial w}{\partial \eta} \right) \right. \\
& \quad \left. + \frac{\partial \log h}{\partial \eta} \cdot \left( \frac{\partial^2}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right) \frac{\partial w}{\partial \eta} \right\} \\
& \quad \quad \quad = \frac{2p \cos^4 \psi}{D} . \tag{4}
\end{aligned}$$

Expressions for the forces and moments in terms of the partials of  $w$  with respect to  $\xi$ ,  $\eta$  coordinates can be obtained from eqs. (1.3:35) and (1.3:38), where  $M_n$ ,  $M_{nt}$  are expressed in terms of partial derivatives of  $w$  with respect to any two orthogonal directions. Such general expressions are very cumbersome in oblique coordinates. But forces and moments expressions on sections  $\xi = \text{const.}$  and  $\eta = \text{const.}$  are as follows:

$$\begin{aligned}
M_\xi &= \text{bending moment per unit length acting on sections } \xi = \text{const.} \\
&= -D \left( \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} \right) \\
&= -\frac{D}{\cos^2 \psi} \left\{ \frac{\partial^2 w}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2 w}{\partial \xi \partial \eta} + (\sin^2 \psi + \sigma \cos^2 \psi) \frac{\partial^2 w}{\partial \eta^2} \right\} . \tag{5}
\end{aligned}$$

$$\begin{aligned}
M_\eta &= \text{bending moment per unit length acting on sections } \eta = \text{const.} \\
&= -D \left( \frac{\partial^2 w}{\partial n^2} + \sigma \frac{\partial^2 w}{\partial z^2} \right) \quad \text{where } n \text{ is the normal to the section } \eta = \text{const.}
\end{aligned}$$

$$\begin{aligned}
\text{Now} \quad x &= \xi \cos \psi - n \sin \psi , \\
y &= \xi \sin \psi + n \cos \psi , \\
\therefore \quad \frac{\partial}{\partial n} &= \frac{\partial x}{\partial n} \frac{\partial}{\partial x} + \frac{\partial y}{\partial n} \frac{\partial}{\partial y} \\
&= \frac{1}{\cos \psi} \left( -\sin \psi \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) .
\end{aligned}$$

Hence

$$M_\eta = -\frac{D}{\cos^2 \psi} \left\{ (\sin^2 \psi + \sigma \cos^2 \psi) \frac{\partial^2 w}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2} \right\} .$$



$M_{\xi t}$  = twisting moment per unit length acting on a section  $\xi = \text{const.}$   
in direction of the normal to the section

$$= D(1-\sigma) \frac{\partial^2 w}{\partial x \partial y}$$

$$= \frac{D(1-\sigma)}{\cos \psi} \left( \frac{\partial^2 w}{\partial \xi \partial \eta} - \sin \psi \frac{\partial^2 w}{\partial \eta^2} \right) \quad (7)$$

$M_{\eta t}$  = twisting moment per unit length acting on a section  $\eta = \text{const.}$   
in direction of the normal to the section

$$= -D(1-\sigma) \frac{\partial^2 w}{\partial \xi \partial \eta}$$

$$= -\frac{D(1-\sigma)}{\cos \psi} \left( -\sin \psi \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \xi \partial \eta} \right) \quad (8)$$

$Q_{\xi}$  = Vertical (in  $Z$  direction) shearing force per unit length acting  
on a section  $\xi = \text{const.}$

$$= -D \frac{1}{\cos^3 \psi} \left( \frac{\partial}{\partial \xi} - \sin \psi \frac{\partial}{\partial \eta} \right) \left( \frac{\partial^2 w}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2} \right)$$

$$= -\frac{D}{\cos^3 \psi} \left\{ \frac{\partial^3 w}{\partial \xi^3} - 3 \sin \psi \frac{\partial^3 w}{\partial \xi^2 \partial \eta} + (1 + 2 \sin^2 \psi) \frac{\partial^3 w}{\partial \xi \partial \eta^2} - \sin \psi \frac{\partial^3 w}{\partial \eta^3} \right\} \quad (9)$$

$Q_{\eta}$  = Vertical shearing force (in  $Z$  direction) per unit length acting  
on the section  $\eta = \text{const.}$

$$= -\frac{D}{\cos^3 \psi} \left( -\sin \psi \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left( \frac{\partial^2 w}{\partial \xi^2} - 2 \sin \psi \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2} \right)$$

$$= -\frac{D}{\cos^3 \psi} \left\{ -\sin \psi \frac{\partial^3 w}{\partial \xi^3} + (1 + 2 \sin^2 \psi) \frac{\partial^3 w}{\partial \xi^2 \partial \eta} - 3 \sin \psi \frac{\partial^3 w}{\partial \xi \partial \eta^2} + \frac{\partial^3 w}{\partial \eta^3} \right\} \quad (10)$$

The determination of the deflection surface of a plate now consists of integrating equation (3) or (4) with appropriate boundary conditions. Several examples are summarized as follows:

- (1) Built-in edge: For built-in edge,  $w$  and  $\frac{\partial w}{\partial \eta}$  must be zero.

If the edge  $\xi = 0$  is built-in, then

$$\begin{aligned} (w)_{\xi=0} &= 0 \\ \left( \frac{\partial w}{\partial \xi} - \sin \psi \frac{\partial w}{\partial \eta} \right)_{\xi=0} &= 0 \end{aligned} \quad (11)$$

- (2) Simply-supported edge: In this case the deflection  $w$  and the bending moment along the edge must be zero. Appropriate moment expression corresponding to the edge in question should be used.

- (3) Free-edge: On a free edge, bending moment  $M_n$  and the reaction  $Q_t - \partial M_{nt} / \partial t$  along the edge must vanish. The twisting moment  $M_{nt}$  must also vanish at the corner if that corner is unsupported. Proper expression for  $M_n$  are to be chosen from (5) or (6) on edges  $\xi = \text{const.}$  or  $\eta = \text{const.}$  respectively. The parts of resultant reactions contributed from the twisting moment  $M_{nt}$  are:

$$\begin{aligned} Q'_\xi &= - \frac{\partial M_{\xi t}}{\partial \eta} = - \frac{\partial M_{\xi t}}{\partial \eta} \\ &= - \frac{D(1-\sigma)}{\cos \psi} \left\{ \frac{\partial^3 w}{\partial \xi^2 \partial \eta^2} - \sin \psi \frac{\partial^3 w}{\partial \eta^3} \right\}, \\ Q'_\eta &= + \frac{\partial M_{\eta t}}{\partial \xi} \\ &= - \frac{D(1-\sigma)}{\cos \psi} \left( - \sin \psi \frac{\partial^3 w}{\partial \xi^3} + \frac{\partial^3 w}{\partial \xi^2 \partial \eta} \right). \end{aligned}$$

Therefore, on an edge corresponding to  $\xi = a$ , say,

$$\begin{aligned} (Q_\xi + Q'_\xi)_{\xi=a} &= 0 \\ &= - \frac{D}{\cos^3 \psi} \left\{ \frac{\partial^3 w}{\partial \xi^3} - 3 \sin \psi \frac{\partial^3 w}{\partial \xi^2 \partial \eta} \right. \\ &\quad \left. + (2 + \sin^2 \psi - \sigma \cos^2 \psi) \frac{\partial^3 w}{\partial \xi \partial \eta^2} \right. \\ &\quad \left. - \sin \psi [1 + (1-\sigma) \cos^2 \psi] \frac{\partial^3 w}{\partial \eta^3} \right\}_{\xi=a} \end{aligned} \quad (12)$$

And on an edge corresponding to  $\eta = b$ , say,

$$(Q_\eta + Q'_\eta)_{\eta=b} = 0$$

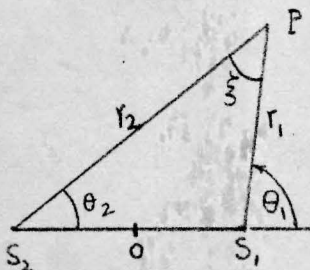
$$= -\frac{D}{\cos^3 \psi} \left\{ -\sin \psi \left[ 1 + (1-\sigma) \cos^2 \psi \right] \frac{\partial^3 w}{\partial \xi^3} + (2 + \sin^2 \psi - \sigma \cos^2 \psi) \frac{\partial^3 w}{\partial \xi^2 \partial \eta} - 3 \sin \psi \frac{\partial^3 w}{\partial \xi \partial \eta^2} + \frac{\partial^3 w}{\partial \eta^3} \right\}_{\eta=b}.$$

(13)

## Appendix I

### 1.5 EQUATIONS OF DEFORMATION IN BIPOLAR COORDINATES.

#### 1.51 INTRODUCTION TO BIPOLAR COORDINATES



Bipolar coordinates may be used to advantage in treating wings with lenticular airfoils (See Ref. 13). Let two circular arcs intersect at two points  $S_1$  and  $S_2$  with rectangular coordinates  $(c, 0)$ ,  $(-c, 0)$  respectively. Consider the conformal transformation

$$z = i c \cot \frac{1}{2} \zeta \quad (1)$$

where  $z = x + iy$ ,  $\zeta = \xi + i\eta$  and  $(x, y)$ ,  $(\xi, \eta)$  are the rectangular coordinates of two corresponding points  $P$  and  $\wp$ . It is easily seen that

$$\frac{z+c}{z-c} = e^{-i\zeta} = e^{\eta - i\xi} \quad (2)$$

Denoting by  $(r_1, \theta_1)$  and by  $(r_2, \theta_2)$  the polar coordinates referred to the points  $S_1$ ,  $S_2$  as origins respectively, we have

$$e^\eta = \frac{r_2}{r_1}, \quad \widehat{S_2 P S_1} = \xi$$

$$r_1^2 = (x-c)^2 + y^2 = |z-c|^2 = 2c M e^{-\eta}, \quad (3)$$

$$r_2^2 = (x+c)^2 + y^2 = |z+c|^2 = 2c M e^\eta,$$

$$\text{where } M = c / (\cosh \eta - \cos \xi) \quad (4)$$



The curves  $\xi = \text{constant}$  are clearly circles through the points  $S_1$  and  $S_2$ , while the curves  $\eta = \text{constant}$  are circles having these points as inverse points. The two sets of curves form in fact two orthogonal systems of circles, as is to be expected since the transformation (1) is conformal.



The value of  $\xi$  varies from  $\xi = 0$  along the x-axis outside of the poles to  $\xi = \pi$  over a flat arc through the poles. The arcs below the poles belong to values of  $\xi$  lying between  $\pi$  and  $2\pi$ . The expressions for  $x$  and  $y$  in terms of  $\xi$  and  $\eta$  are

$$\begin{aligned} x &= M \sinh \eta \\ y &= M \sin \xi \end{aligned} \quad (5)$$

At a point  $P_0$  of the line  $S_1 S_2$  we have  $\xi = \pi$ , therefore

$$x_0 = c \tanh \frac{\eta_0}{2}, \quad y_0 = 0 \quad (6)$$

The first fundamental differential form is:

$$d\ell^2 = \frac{c^2}{(\cosh \eta - \cos \xi)^2} [d\xi^2 + d\eta^2] \quad (7)$$

## 1.52 COORDINATE SYSTEM AND DERIVATION OF ELASTIC DEFORMATION EQUATIONS.

Let us take an orthogonal coordinate system  $\xi, \eta, \zeta$  where  $\eta$  and  $\zeta$  form a bipolar coordinate system. The line element  $d\ell$  is expressed in this system by

$$d\ell^2 = d\xi^2 + \frac{c^2}{(\cosh \eta - \cos \zeta)^2} (d\eta^2 + d\zeta^2) \quad (1)$$

$$= d\xi^2 + M^2 (d\eta^2 + d\zeta^2)$$

$$\text{where } M = \frac{c}{\cosh \eta - \cos \zeta} \quad (2)$$

The poles of the bipolar coordinates, are therefore situated at distances  $\pm c$  from the  $\xi$  axis.

Let a lenticular airfoil be given with upper and lower camber corresponding to  $\xi = \pi \mp \epsilon$  respectively. Let  $\bar{s}$  (with components  $s_\xi, s_\eta, s_\zeta$ ) be the elastic displacement vector within the plate, and let  $e_{ij}$  and  $p_{ij}$  (with components  $e_{\xi\xi}, e_{\xi\eta}, p_{\xi\xi}, p_{\xi\eta}$  etc.) be again the strain and stress tensors respectively.

As before the boundary conditions on the faces of the plate when it is subject to distributed tractions normal to the faces are

$$\begin{aligned} p_{\xi\xi}(\pi \pm \epsilon) &= \pm p, \\ p_{\xi\xi}(\pi \pm \epsilon) &= p_{\xi\eta}(\pi \pm \epsilon) = 0. \end{aligned} \quad (3)$$

This can also be written in expanded form as

$$\begin{aligned} p_{\xi\xi}(\pi) + \frac{1}{2!} \epsilon^2 p_{\xi\xi}''(\pi) + \dots &= 0, \\ p_{\xi\xi}'(\pi) + \frac{1}{3!} \epsilon^2 p_{\xi\xi}'''(\pi) + \dots &= \frac{p}{\epsilon}, \\ p_{\xi\xi}(\pi) + \frac{1}{2!} \epsilon^2 p_{\xi\xi}''(\pi) + \dots &= 0, \\ p_{\xi\xi}'(\pi) + \frac{1}{3!} \epsilon^2 p_{\xi\xi}'''(\pi) + \dots &= 0, \end{aligned} \quad (4)$$

etc.

Let  $A$  be defined by dq. (1.3:6), and let two fictitious stress components  $q_{\xi\xi}$ ,  $q_{\eta\eta}$  be defined by the relations:

$$p_{\xi\xi} = (A-1)p_{\xi\xi} + q_{\xi\xi}, \quad p_{\eta\eta} = (A-1)p_{\xi\xi} + q_{\eta\eta}. \quad (5)$$

Whence

$$\begin{aligned} q_{\xi\xi} &= 2\mu \left[ A e_{\xi\xi} + (A-1)e_{\eta\eta} \right], \\ q_{\eta\eta} &= 2\mu \left[ (A-1)e_{\xi\xi} + A e_{\eta\eta} \right]. \end{aligned} \quad (6)$$

From the result of tensor analysis (see, for example, Ref. 12),

$$\begin{aligned} e_{\xi\xi} &= \frac{\partial S_\xi}{\partial \xi}, \\ e_{\eta\eta} &= \frac{\partial}{\partial \eta} \left[ (\cosh \eta - \cos \zeta) \frac{S_\eta}{c} \right] - \left[ \frac{\sinh \eta}{c} S_\eta + \frac{\sin \zeta}{c} S_\zeta \right], \\ e_{\xi\xi} &= \frac{\partial}{\partial \xi} \left[ (\cosh \eta - \cos \zeta) \frac{S_\xi}{c} \right] - \left[ \frac{\sinh \eta}{c} S_\eta + \frac{\sin \zeta}{c} S_\zeta \right]. \end{aligned} \quad (7)$$

Therefore

$$\begin{aligned} q_{\xi\xi} &= 2\mu \left\{ A \frac{\partial S_\xi}{\partial \xi} + (A-1) \frac{\partial}{\partial \eta} \left( \frac{S_\eta}{M} \right) - \frac{(A-1)}{c} [\sinh \eta S_\eta + \sin \zeta S_\zeta] \right\}, \\ q_{\eta\eta} &= 2\mu \left\{ (A-1) \frac{\partial S_\xi}{\partial \xi} + A \frac{\partial}{\partial \eta} \left( \frac{S_\eta}{M} \right) - \frac{A}{c} (\sinh \eta S_\eta + \sin \zeta S_\zeta) \right\}. \end{aligned} \quad (8)$$

Other stress components, expressed in terms of displacements, are (cf. eq. 1.2:2)

$$\frac{1}{\mu} p_{\xi\xi} = \nu \left\{ \frac{\partial s_{\xi}}{\partial \xi} + \frac{1}{M} \left( \frac{\partial s_{\eta}}{\partial \eta} + \frac{\partial s_{\xi}}{\partial \xi} \right) - \frac{\gamma+2}{rc} (\sinh \eta s_{\eta} + \sin \xi s_{\xi}) \right\} + 2 \frac{\partial}{\partial \xi} \left( \frac{s_{\xi}}{M} \right),$$

$$\frac{1}{\mu} p_{\xi\xi} = \frac{1}{M} \frac{\partial s_{\xi}}{\partial \xi} + \frac{\partial s_{\xi}}{\partial \xi},$$

$$\frac{1}{\mu} p_{\xi\eta} = \frac{1}{M} \left( \frac{\partial s_{\eta}}{\partial \xi} + \frac{\partial s_{\xi}}{\partial \eta} \right) + \frac{1}{c} (s_{\eta} \sin \xi + s_{\xi} \sinh \eta),$$

$$\frac{1}{\mu} p_{\eta\xi} = \frac{1}{M} \frac{\partial s_{\xi}}{\partial \eta} + \frac{\partial s_{\eta}}{\partial \xi}. \quad (9)$$

The equations of equilibrium, written in terms of stress components, take the following form: (cf. eq. (1.2.1))

$$\begin{aligned} \frac{\partial}{\partial \xi} (M^2 p_{\xi\xi}) + \frac{\partial}{\partial \eta} (M p_{\xi\eta}) + \frac{\partial}{\partial \xi} (M p_{\xi\xi}) &= 0 \\ \frac{\partial}{\partial \eta} (M p_{\eta\eta}) + \frac{\partial}{\partial \xi} (M p_{\eta\xi}) + \frac{\partial}{\partial \xi} (M^2 p_{\eta\xi}) + \frac{\partial M}{\partial \xi} p_{\eta\xi} - \frac{\partial M}{\partial \eta} p_{\xi\xi} &= 0 \\ \frac{\partial}{\partial \xi} (M p_{\xi\xi}) + \frac{\partial}{\partial \xi} (M^2 p_{\xi\xi}) + \frac{\partial}{\partial \eta} (M p_{\xi\eta}) + \frac{\partial M}{\partial \eta} p_{\xi\eta} - \frac{\partial M}{\partial \xi} p_{\eta\eta} &= 0 \end{aligned} \quad (10)$$

Eliminating  $p_{\xi\xi}$ ,  $p_{\xi\xi}$ ,  $p_{\xi\eta}$  with the help of eq. (3), and observing that  $\frac{\partial M}{\partial \xi}$  vanishes when  $\xi = \pi$ , we find the equation:

$$\frac{\partial}{\partial \xi} (M^2 p_{\xi\xi}''') + \frac{\partial}{\partial \eta} (M p_{\xi\eta}''') + \frac{\partial M}{\partial \eta} p_{\xi\eta}'' + \frac{M}{3} p_{\xi\xi}''' = \frac{2P}{\epsilon^3}, \quad (11)$$

where all the quantities are to be evaluated at  $\xi = \pi$ . This is the differential equation of deflection of the plate, in first approximation. To calculate the terms of eq. (11), with accuracy to zero order in  $\epsilon$ , we proceed as follows: From eqs. (9) and (4), we find, (for  $\xi = \pi$ ),



$$s'_5 = \frac{(A-1)c}{\cosh \eta + 1} \left\{ -\frac{\partial s_5}{\partial \xi} - \frac{\cosh \eta + 1}{c} \frac{\partial s_\eta}{\partial \eta} + \frac{1}{c(A-1)} \sinh \eta s_\eta \right\},$$

$$s'_3 = -\frac{c}{\cosh \eta + 1} \frac{\partial s_5}{\partial \xi}, \quad (12)$$

$$s'_\eta = -\frac{\partial s_5}{\partial \eta} - \frac{\sinh \eta}{\cosh \eta + 1} s_5.$$

Similarly, differentiate (9) with respect to  $\xi$  and apply the conditions (4), we find, when (12) is taken into account,

$$s''_5 = \frac{(A-1)c}{\cosh \eta + 1} \left\{ \frac{p}{\gamma \mu c} + \frac{c}{\cosh \eta + 1} \frac{\partial^2 s_5}{\partial \xi^2} + \frac{\cosh \eta + 1}{c} \frac{\partial}{\partial \eta} \left( \frac{\sinh \eta}{\cosh \eta + 1} s_5 \right) \right. \\ \left. - \frac{\gamma + 2}{\gamma c} \sinh \eta \left( \frac{\partial s_5}{\partial \eta} + \frac{\sinh \eta}{\cosh \eta + 1} s_5 \right) + \frac{2A-3}{A-1} \frac{s_5}{c} \right\},$$

$$s''_3 = -\frac{c}{\cosh \eta + 1} \frac{\partial s'_5}{\partial \xi},$$

$$s''_\eta = \frac{1}{\cosh \eta + 1} (\sinh \eta s'_5 - s_\eta) - \frac{\partial s'_5}{\partial \eta}. \quad (13)$$

From (8), (9), (12) and (13) we obtain

$$\frac{1}{2\mu} q'_{\xi\xi} = -AM \frac{\partial^2 s_5}{\partial \xi^2} + (A-1) \left\{ \frac{1}{M} \frac{\partial}{\partial \eta} \left[ -\frac{\partial s_5}{\partial \eta} - \frac{\sinh \eta}{\cosh \eta + 1} s_5 \right] + \frac{s_5}{c} \right\},$$

$$\frac{1}{2\mu} q''_{\xi\xi} = -AM \frac{\partial^2 s'_5}{\partial \xi^2} + (A-1) \left\{ \frac{s'_5}{M} + \frac{2 \sinh \eta}{c} \frac{\partial s'_5}{\partial \eta} - \frac{2}{c} \frac{\partial s_\eta}{\partial \eta} - \frac{1}{M} \frac{\partial^2 s'_5}{\partial \eta^2} \right. \\ \left. - \frac{\sinh \eta}{c(\cosh \eta + 1)} (\sinh \eta s'_5 - s_\eta) \right\},$$

$$\frac{1}{2\mu} q'_{\eta\eta} = -(A-1)M \frac{\partial^2 s_5}{\partial \xi^2} + A \left\{ \frac{1}{M} \frac{\partial}{\partial \eta} \left[ -\frac{\partial s_5}{\partial \eta} - \frac{\sinh \eta}{\cosh \eta + 1} s_5 \right] + \frac{s_5}{c} \right\},$$

$$\frac{1}{2\mu} q_{\eta\eta}'' = -(A-1)M \frac{\partial^2 s_s'}{\partial \xi^2} + A \left\{ \frac{s_s'}{M} + 2 \frac{\sinh \eta}{c} \frac{\partial s_s'}{\partial \eta} - \frac{2}{c} \frac{\partial s_\eta}{\partial \eta} - M \frac{\partial^2 s_s'}{\partial \eta^2} - \frac{\sinh \eta}{c(\cosh \eta + 1)} (\sinh \eta s_s' - s_\eta) \right\},$$

$$\frac{1}{2\mu} p_{\eta\xi}' = - \frac{\partial^2 s_s}{\partial \xi \partial \eta}$$

$$\frac{1}{2\mu} p_{\eta\xi}'' = - \frac{\partial^2 s_s'}{\partial \xi \partial \eta} + \frac{\sinh \eta}{\cosh \eta + 1} \frac{\partial s_s'}{\partial \xi} - \frac{1}{2c} \frac{\partial s_s}{\partial \eta} - \frac{1}{2(\cosh \eta + 1)} \frac{\partial s_\eta}{\partial \xi}. \quad (14)$$

Taking into account the relations (5), the field equations (10) acquire in bipolar coordinates the form

$$\begin{aligned} M p_{\xi\xi}' &= - \frac{\partial}{\partial \xi} (M^2 q_{\xi\xi}) - \frac{\partial}{\partial \eta} (M p_{\xi\eta}) - \frac{\partial M}{\partial \xi} p_{\xi\xi} - (A-1) M^2 \frac{\partial}{\partial \xi} p_{\xi\xi}, \\ M p_{\xi\eta}' &= - \left[ \frac{\partial M}{\partial \eta} (A-2) + M(A-1) \frac{\partial}{\partial \eta} \right] p_{\xi\xi} - \frac{\partial M}{\partial \eta} q_{\eta\eta} \\ &\quad - M \frac{\partial q_{\eta\eta}}{\partial \eta} - \frac{\partial}{\partial \xi} (M^2 p_{\eta\xi}) - 2 \frac{\partial M}{\partial \xi} p_{\eta\xi}, \end{aligned} \quad (15)$$

$$M p_{\xi\xi}' = \frac{\partial M}{\partial \xi} (A-2) p_{\xi\xi} - \frac{\partial}{\partial \xi} (M^2 p_{\xi\xi}) - M \frac{\partial p_{\xi\eta}}{\partial \eta} - 2 \frac{\partial M}{\partial \eta} p_{\xi\eta} + \frac{\partial M}{\partial \xi} q_{\eta\eta}.$$

Differentiating (15) once and twice with respect to  $\xi$ , and set  $\xi = \pi$ , we obtain

$$M p_{\xi\xi}'' = - \frac{\partial}{\partial \xi} (M^2 q_{\xi\xi}') - \frac{\partial}{\partial \eta} (M p_{\xi\eta}') - \frac{\partial^2 M}{\partial \xi^2} p_{\xi\xi} - (A-1) M^2 \frac{\partial}{\partial \xi} p_{\xi\xi}',$$

$$\begin{aligned} M p_{\xi\eta}'' &= - \left[ \frac{\partial M}{\partial \eta} (A-2) + M(A-1) \frac{\partial}{\partial \eta} \right] p_{\xi\xi}' - \frac{\partial M}{\partial \eta} q_{\eta\eta}' - M \frac{\partial q_{\eta\eta}'}{\partial \eta} \\ &\quad - \frac{\partial}{\partial \xi} (M^2 p_{\eta\xi}') - 2 \frac{\partial^2 M}{\partial \xi^2} p_{\eta\xi}, \end{aligned}$$

$$M p_{\xi\xi}'' = \frac{\partial^2 M}{\partial \xi^2} (A-2) p_{\xi\xi} - \frac{\partial}{\partial \xi} (M^2 p_{\xi\xi}') - M \frac{\partial p_{\xi\eta}}{\partial \eta} - 2 \frac{\partial M}{\partial \eta} p_{\xi\eta}' + \frac{\partial^2 M}{\partial \xi^2} q_{\eta\eta},$$

$$\begin{aligned}
M p_{55}''' = & -\frac{\partial^2 M}{\partial \xi^2} p_{55}' + 2 \frac{\partial^2 M}{\partial \xi^2} (A-2) p_{55}' - M^2 \frac{\partial^2 p_{55}''}{\partial \xi} - 2M \frac{\partial^2 M}{\partial \xi^2} \frac{\partial p_{55}}{\partial \xi} \\
& - M \frac{\partial^2 p_{5\eta}''}{\partial \eta} - \frac{\partial^2 M}{\partial \xi^2} \frac{\partial p_{5\eta}}{\partial \eta} - 2 \frac{\partial M}{\partial \eta} p_{5\eta}'' - 2 \frac{\partial^2 M}{\partial \eta \partial \xi^2} p_{5\eta} + 2 \frac{\partial^2 M}{\partial \xi^2} q_{\eta\eta}' .
\end{aligned}
\tag{16}$$

Substituting the last equation of (16) into (11) we obtain, since  $\frac{\partial M}{\partial \xi}, \frac{\partial^3 M}{\partial \xi^3}$  vanish when  $\xi = \pi$ ,

$$\begin{aligned}
2 M^2 \frac{\partial^2 p_{55}''}{\partial \xi} + 2 M \frac{\partial p_{5\eta}''}{\partial \eta} + 4 \frac{\partial M}{\partial \eta} p_{5\eta}'' + (2A-5) \frac{\partial^2 M}{\partial \xi^2} p_{55}' - 2M \frac{\partial^2 M}{\partial \xi^2} \frac{\partial p_{55}}{\partial \xi} \\
- \frac{\partial^2 M}{\partial \xi^2} \frac{\partial p_{5\eta}}{\partial \eta} - 2 \frac{\partial^2 M}{\partial \eta \partial \xi^2} p_{5\eta} + 2 \frac{\partial^2 M}{\partial \xi^2} q_{\eta\eta}' = \frac{6P}{\epsilon^3} .
\end{aligned}
\tag{17}$$

From which we deduce, using (4), (16), and (15),

$$\begin{aligned}
-M^3 \frac{\partial^2}{\partial \xi^2} q_{\xi\xi}' - 2 M^2 \frac{\partial^2 p_{5\eta}'}{\partial \eta \partial \xi} - M \frac{\partial^2 q_{\eta\eta}'}{\partial \eta^2} - \frac{1}{M} \left( \frac{\partial M}{\partial \eta} \right)^2 q_{\eta\eta}' - 3 \frac{\partial M}{\partial \eta} \frac{\partial q_{\eta\eta}'}{\partial \eta} \\
- 4 M \frac{\partial M}{\partial \eta} \frac{\partial p_{\eta\xi}'}{\partial \xi} = \frac{3P}{\epsilon^3} .
\end{aligned}
\tag{18}$$

Then, substituting (14) into (18), we obtain finally

$$\begin{aligned}
\frac{\partial^4 W}{\partial \xi^4} + 2 \left( \frac{\cosh \eta + 1}{c} \right)^2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{A-1}{Ac^2} (5 \sinh^2 \eta - \cosh \eta - 1) \frac{\partial^4 W}{\partial \xi^2} \\
- \frac{4}{c^2} \sinh \eta (\cosh \eta + 1) \frac{\partial^3 W}{\partial \xi^2 \partial \eta} + \left( \frac{\cosh \eta + 1}{c} \right)^4 \left( 1 + \frac{\sinh \eta}{\cosh \eta + 1} \right) \frac{\partial^3 W}{\partial \eta^3} \\
+ \frac{1}{c^4} \left\{ (\cosh \eta + 1)^3 (3 \cosh \eta - \sinh \eta - 1) - 4 (\cosh \eta + 1)^2 \sinh^2 \eta \right\} \frac{\partial^2 W}{\partial \eta^2} \\
+ \left\{ 3 (\cosh \eta + 1)^3 (\sinh \eta + \cosh \eta) + (\cosh \eta + 1)^2 (10 \sinh \eta \cosh \eta \right. \\
\left. - 2 \sinh^2 \eta + 3 \sinh \eta) + 6 (\cosh \eta + 1) \sinh^3 \eta \right\} \frac{\partial W}{\partial \eta} \\
+ \left\{ \cosh \eta (\cosh \eta + 1)^3 - 3 (\cosh \eta + 1)^2 \sinh \eta \cosh \eta + (\cosh^2 \eta - 1) \sinh^2 \eta \right. \\
\left. - \sinh^4 \eta \right\} W = \frac{3P}{2 A \mu \epsilon^3 M^4} .
\end{aligned}
\tag{19}$$



## II

### THE PLATE PROBLEM AS A VARIATIONAL PROBLEM

#### 2.1 EQUIVALENCE BETWEEN A DIFFERENTIAL SYSTEM AND A VARIATIONAL PROBLEM.

It is well known that a boundary value problem of partial differential equation of mathematical physics can usually be replaced by an equivalent variational problem.\* This equivalence is of great importance because direct methods of solving the variational problem sometimes offer powerful means to obtain useful approximate solutions.

The well-known Kirchhoff's theory of plates utilizes the principle of minimum energy and the variational principle. (See Ref. 2) Starting from the assumption that the effect of shearing stresses on the deflection of a plate is negligible, he obtained an expression of total strain energy of the bent plate, which contains only terms depending on the action of bending and twisting moments, as follows:

$$U = \frac{1}{2} D \iint_R \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\sigma) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy, \quad (1)$$

Where the integration is extended over the entire region  $R$  of the mid-surface of the plate. The potential energy of the external forces and moments are given by

$$W = \iint_R q w dx dy - \int_L M_n \frac{\partial w}{\partial n} ds + \int_L \left( Q_n - \frac{\partial M_{nt}}{\partial s} \right) w ds, \quad (2)$$

where the line integral is integrated in the positive sense (so that  $n$  is to  $s$  as  $x$  is to  $y$  axis) along the boundary curve  $L$  of the region  $R$ . " $n$ " is the outer normal.

The problem is then to find a unique function  $w$ , which minimizes the function

$$V = U - W, \quad (3)$$

with the auxiliary condition of continuity on  $w$  and in conformity with the conditions of support at the edges  $L$  of the plate.

A necessary condition that  $V(w)$  reaches a minimum is that the first variation  $\delta V(w)$  vanishes. Direct calculation gives

$$\delta V = \iint_R (D \nabla^4 w - q) \delta w dx dy + \quad (4)$$

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\* A rigorous formulation of this equivalence is given by Courant, Ref 25, Vol. II, chapter VII.

$$\begin{aligned}
& + \int_L \left\{ D \left[ (1-\sigma) \left( \frac{\partial^2 w}{\partial x^2} \cos^2 \theta + \sin 2\theta \frac{\partial^2 w}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 w}{\partial y^2} \right) + \sigma \nabla^2 w \right] + \right. \\
& \quad \left. + M_n \left\{ \frac{\partial \delta w}{\partial n} ds + \right. \right. \\
& \quad \left. + \int_L \left\{ D \left[ (1-\sigma) \frac{\partial}{\partial s} \left\{ \frac{1}{2} \sin 2\theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right\} - \right. \right. \\
& \quad \left. \left. - \cos \theta \nabla^2 \frac{\partial w}{\partial x} - \sin \theta \nabla^2 \frac{\partial w}{\partial y} \right] - \left( Q_n - \frac{\partial M_n}{\partial s} \right) \right\} \delta w ds.
\end{aligned}$$

where  $\delta w$  and  $\frac{\partial \delta w}{\partial n}$  are arbitrary small quantities satisfying the prescribed conditions of displacements on the boundary  $L$ , and  $\theta$  is the angle between the outer normal to the boundary and the  $x$ -axis. Since  $\delta w$  and  $\frac{\partial \delta w}{\partial n}$  are arbitrary,  $\delta V$  vanishes if and only if each of the three integrals in (4) vanishes. The first integral vanishes if and only if

$$\nabla^4 w = \frac{q}{D} \quad (5)$$

for every point in  $R$ . This equation, the Euler equation of the variational problem, is the field equation governing the deformation of the plate (cf. eq. 1.3:28). Therefore, we may say that the form of the strain energy expression adopted, eq. (1), is compatible with the first approximation of the procedure in Chapter I, and the Poisson-Love theory.

The vanishing of the other two integrals in (4) gives us the natural boundary conditions.

If an edge of a plate is built-in, where  $\delta w$  and  $\frac{\partial \delta w}{\partial n}$  as well as  $w$  and  $\frac{\partial w}{\partial n}$  are zero, then both line integrals vanish identically. If, on the other hand, an edge  $L_i$  of the plate is simply-supported, where  $\delta w = w = 0$  and  $M_n = 0$ , then the last integral in (4) is zero but the second integral vanishes only if

$$(1-\sigma) \left( \frac{\partial^2 w}{\partial x^2} \cos^2 \theta + \frac{\partial^2 w}{\partial x \partial y} \sin 2\theta + \frac{\partial^2 w}{\partial y^2} \sin^2 \theta \right) + \sigma \nabla^2 w = 0 \quad (6)$$

on  $L_i$ .

Further, if an edge  $L_i$  of the plate is unsupported and free from any external forces, the quantities  $\delta w$  and  $\frac{\partial \delta w}{\partial n}$  are entirely arbitrary on  $L_i$  but  $M_n = 0$  and  $Q_n - \frac{\partial M_n}{\partial s} = 0$  on  $L_i$ . Then we must have the boundary conditions on  $L_i$ :

$$\begin{aligned}
(1-\sigma) \left( \frac{\partial^2 w}{\partial x^2} \cos^2 \theta + \frac{\partial^2 w}{\partial x \partial y} \sin 2\theta + \frac{\partial^2 w}{\partial y^2} \sin^2 \theta \right) + \sigma \nabla^2 w &= 0, \\
(1-\sigma) \frac{\partial}{\partial s} \left[ \frac{1}{2} \sin 2\theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right] - \\
- \cos \theta \nabla^2 \frac{\partial w}{\partial x} - \sin \theta \nabla^2 \frac{\partial w}{\partial y} &= 0.
\end{aligned} \quad (7)$$

These conditions agree with those stated in Chapter I.

In case a given moment  $M_n$  and a transverse force  $Q_n - \frac{\partial M_{nt}}{\partial s}$  are distributed along a supported or unsupported edge of the plate, corresponding boundary conditions can again be formulated from the condition  $\delta V = 0$ . They are, in fact, obtained by adding the terms  $\frac{1}{\delta} M_n$  and  $-\frac{1}{\delta} (Q_n - \frac{\partial M_{nt}}{\partial s})$  to the left hand side of the 1st and 2nd of equations (7) respectively.

This example illustrates clearly the way of obtaining the fundamental partial differential equation of deformation and appropriate boundary conditions from the variational principle. Indeed, the principle of minimum energy is such a fundamental concept in physics that we may take it as a starting point for our discussion. In practice, however, we are confronted with the problem of forming a proper expression for the potential energy to begin with. The expression (1) is found satisfactory as a first approximation. But what are the proper expressions of  $U$  in higher order approximations? For a satisfactory theory we must be able to derive systematically the proper expressions of strain energy of a bent plate to any order of approximation without imposing arbitrary assumptions regarding to the deformation of the plate. In all events it seems that this problem is as complicated and is of the same nature as the problem of the last chapter.

The equivalence between a boundary value problem of partial differential equation and the corresponding variational problem can be looked from another point of view. Suppose we have obtained a differential system for the deformation of a plate by certain process of reasoning. But instead of proceeding as usual from the general solution to the particular solution, which oft-times is either impractical or impossible, we like to set up the problem in a variational form so that useful approximate solutions can be obtained by some direct methods of approximation. Thus if we are given with a differential system consists of eq. (5) together with the boundary conditions (6) or (7), we may set up the problem as to find a solution of the equation  $\delta V = 0$ , where  $\delta V$  is given by (4). In fact this is only a restatement of the given problem, but in this form the direct methods of approximations (to be enumerated in the next two sections) are applicable.

In slightly more general terms, if a differential system consists of a differential equation  $D(w) = 0$  over a region  $R$  and boundary conditions  $B_i(w) = 0, (i=1, 2, \dots)$  on the boundary curve  $L$  of  $R$ , we may represent the given problem as to find a function  $w$ , so that

$$\iint_R D(w) \delta w \, dx \, dy + \sum_{i=1, 2, \dots} \int_L B_i(w) \delta_i w \, ds = 0 \quad (8)$$

where  $\delta_i w$  are perfectly arbitrary functions.



If we can find an integral  $\iint_R E(w) dx dy$  of which the vanishing of the first variation leads to the above equation (8), then we can write (8) as  $\delta \iint_R E(w) dx dy = 0$ . Such an integral in general has a simple physical meaning, as usually is an energy integral. But when we start from a given differential system, it is not at all evident that such an integral can be found easily. Our formulation of the equivalent variational equation (8) therefore avoids this difficulty.

There is another reason for not attaching too much importance on finding the energy integral. This is based on 'convergency' considerations. In direct methods we approach the true solution  $w$  by forming a sequence of approximating functions  $w_N$ , ( $N = 1, 2, 3, \dots$ ), and by passing  $N$  to infinity to get the required solution. But sometimes when an approximating sequence  $w_N$  is found so that the integral

$\iint_R E(w_N) dx dy$  readily converges to the true minimum value of

$\iint_R E(w) dx dy$ , the function  $w$  itself, or its derivatives, may not be convergent. An integration has a smoothing effect. In general, the convergency of  $w_N$  or its derivatives is improved if the order of the derivatives of  $w$  occurring in the integrand  $E(w)$  becomes higher. For example, Courant remarked (Ref. 14) that it is really a credit to Walther Ritz for his spectacular success in his direct procedure in having chosen the problem of vibration of plates for illustration instead of taking the seemingly easier corresponding problem of membrane. For in the latter, the convergency will be very poor. Courant pointed out (Ref. 14) that the convergency can be improved by adding to the original minimizing integral expressions which consist of higher order derivatives of  $w$  and which vanish for the true solution. For example, we may formulate the plate problem as to find a function  $w$  which minimizes the function

$$V'(w) = U - W + k \iint_R (\nabla^2 w - \frac{q}{D})^2 dx dy, \quad (9)$$

where  $U$  and  $W$  are given by (1) and (2) respectively and " $k$ " is any positive constant or positive function of  $(x, y)$ . Such additional terms make  $V(w)$  more sensitive to variations of  $w$  without changing the true solution. In other words, the minimizing sequence attached to such a "sensitized" functional will by force behave better as regards convergence.

But the "sensitization" spoils the physical meaning of the original integral. So again we feel no need to stress too much on the importance of the physical meaning of the minimizing integral.

Equation (8) is certainly not the only way to represent a differential system  $D(w) = 0$  in a region  $R$  and  $B_i(w) = 0$ , ( $i = 1, 2, \dots$ ) on the boundary  $L$  of  $R$ . We may as well represent it as

$$\iint_R D^2(w) dx dy + \sum_i \int_L B_i^2(w) ds = 0. \quad (10)$$

In this form we interpret  $D(w_N)$ ,  $B_i(w_N)$ , ( $i=1, 2, \dots$ ) where  $w_N$  is an approximating functional sequence of  $w$ , as error functions. We may state our problem as to find a sequence  $w_N$ , so that the error functions

$$\iint_R D^2(w_N) dx dy + \sum_i \int_L B_i^2(w_N) ds$$

tend to a minimum value of 0 when  $N$  tends to infinity.

The classical method of solving a variational problem by forming its Euler's equation and then solving it with appropriate boundary conditions is an indirect method. The alternative idea of looking for a single sequence of functions which solves the given minimal problem is called the direct method. The direct approach was first conceived by J. Bernoulli and later by Riemann (Ref. 15). Through the fundamental work of Hilbert (Ref. 16) it was revived and developed fruitfully with a new-born impulse. The utilization of this method to numerical calculations of the solutions was independently envisaged by two great physicists Lord Rayleigh (Ref. 5) and Walther Ritz (Ref. 17, 18). Since then numerous applications and improvements have been made by many authors. (see Refs. 10, 11, and 19).

To state in general terms, our problem is the following: We are given an integral expression  $V[\varphi]$  over a given domain  $R$  of the independent variables, whose boundary satisfies all the desired continuity assumptions, and we want to find a function  $\varphi = w$  for which  $V[w] = d$ , the lower limit of  $V[\varphi]$  for the totality of all  $\varphi$ 's which satisfy the continuity conditions and boundary conditions. We assume that such a lower limit exists, and that the integrand satisfies all the continuity and regularity conditions. The assumption on the definability implies the existence of a so-called minimal sequence  $\varphi_1, \varphi_2, \varphi_3, \dots$  of allowable functions, for which

$$\lim_{N \rightarrow \infty} V[\varphi_N] = d.$$

The fundamental conception of all direct methods is such, that each  $\varphi_N$  be obtained from a definite elementary minimal problem, and through the passing to limit  $N \rightarrow \infty$  to get the required solution  $w$  of the given minimal problem. These two principal steps, the construction of the minimal sequence on basis of common minimal problems and the passing to limit, characterizes the direct methods.

We now proceed to formulate the detail procedures of calculation.



## 2.2 RAYLEIGH-RITZ METHOD AND ITS ALLIED METHODS.

The Rayleigh-Ritz method consists in narrowing down the choice of  $\phi$  (see the last paragraph of 2.1) into a smaller class of functions. We assume that  $\phi$  can be written as functions that satisfy the boundary conditions and involve a number of parameters. These parameters are then determined so as to make  $V(\phi)$  a minimum. Many important modifications to the original Rayleigh-Ritz procedure have been suggested. We denote under the present catalogue those methods which are characterized by adopting as their approximating functions (or coordinate functions) those satisfy the boundary conditions, but in general not the differential equations.

For a first approximation the deflection  $w$  of an elastic plate minimizes the potential energy integral

$$V(w) = \frac{D}{2} \iint_R \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\sigma) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy - \iint_R q w dx dy + \int_L M_n \frac{\partial w}{\partial n} ds - \int_L \left( Q_n - \frac{\partial M_{nt}}{\partial s} \right) w ds. \quad (1)$$

In order to obtain an approximate solution of the plate problem we select from the set of all admissible functions satisfying the prescribed boundary conditions a certain subset. We assume that  $w$  can be represented with sufficient accuracy by a series of the form

$$w_N = \sum_{i=1}^N c_i \Phi_i(x, y) \quad (2)$$

where  $\Phi_i$  are so chosen that  $w_N$  satisfies the same continuity conditions and boundary conditions of  $w$ , but not necessarily the differential equation governing  $w$ . If (2) is inserted for  $w$  into the integral of  $V$ , eq. (1), the latter becomes a function  $V_N$  of the parameters  $c_i$ , ( $i = 1, 2, \dots, N$ ). Since the strain-energy density is a quadratic function of the strains  $\epsilon_{ij}$ , which in turn are linear functions of the derivatives of displacements, it is clear that  $V_N$  is a quadratic function of the parameters  $c_i$ . The minimizing conditions

$$\frac{\partial V_N}{\partial c_i} = 0, \quad (i = 1, 2, \dots, N) \quad (3)$$

provide therefore  $N$  linear equations to determine the  $N$  unknown constants  $c_i$ . The approximate deflection surface  $w_N$  is thus determined. This is the Rayleigh-Ritz procedure. (Refs. 5, 17, 18).



The method of Galerkin (Ref. 20) may be introduced as follows: In 2.1 it was shown that the minimization of the energy integral  $V$  leads to the equation (2.1.4)

$$\iint_R D(w) \delta w \, dx \, dy + \int_L B_1(w) \delta w \, ds + \int_L B_2(w) \frac{\partial \delta w}{\partial n} \, ds = 0, \quad (4)$$

where  $D(w) \equiv \nabla^2 w - \frac{q}{D}, \quad (5)$

and  $B_1(w)$  and  $B_2(w)$  are appropriate boundary conditions on  $L$ ,  $\delta w$  and  $\partial \delta w / \partial n$  are arbitrary unless restricted by edge constraint. Now if again we take  $w_N$  in the form of the series (2) where  $w_N$  satisfies the boundary conditions imposed on  $w$ , then the last two integrals of (4) vanish automatically for  $w = w_N$ . When  $w_N$  is substituted into Eq. (4), which now becomes

$$\iint_R D(w) \delta w \, dx \, dy = 0, \quad (6)$$

it will not be satisfied for arbitrary variations  $\delta w$ , since this would imply that  $w_N(x, y)$  satisfies the Euler equation  $D(w) = 0$  and hence is an exact solution. Although the variation in  $V$  does not now vanish for an arbitrary variation  $\delta w$ , yet the constants  $c_i$  in (2) can be so chosen that the variation in  $V$  is equal to zero for a set of  $N$  values of  $\delta w$ , which we now take as  $k_i \varphi_i(x, y)$ , where  $\varphi_i(x, y)$ ,  $i = 1, 2, \dots, N$  are the coordinate functions of  $w_N$ , as defined by eq. (2), and  $k_i$  are positive constants. If, now,  $N$  becomes infinite and the functions  $\varphi_i(x, y)$  form a complete system\* of functions, then the set of all relations

$$\iint_R D(w_N) \varphi_i \, dx \, dy = 0, \quad (i = 1, 2, \dots, N), \quad (7)$$

becomes equivalent to the relation (6). This method of determining  $c_i$  is the Galerkin's method. It proceeds directly from the differential equation  $D(w) = 0$  to equation (7).

A generalization due to L. V. Kantorovitch (Ref. 21) is the following. In obtaining an approximate solution of  $D(w) = 0$  in the form of (2), the constants  $c_i$  are replaced by unknown functions of one variable  $c_i(x)$ , say, and an application of the minimum principle leads to a system of ordinary differential equations for the functions  $c_i(x)$ .

The collocation method introduced by Biezeno and Koch (Ref. 22, 23) consists in another interpretation of the Eq. (6). Instead of requiring that (6) be true when  $\delta w$  takes the value of a set of  $N$

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\*A set of functions  $f_i(x)$  is said to form a complete system if any piecewise continuous function  $f(x)$  can be approximated by a sum  $\sum_{i=1}^N c_i f_i(x)$  in such a way that the mean square error  $\int [f(x) - \sum_{i=1}^N c_i f_i(x)]^2 dx$  can be made arbitrarily small by a suitable choice of  $N$ . (Ref. 25)

functions  $\varphi_i$ , ( $i = 1, 2, \dots, N$ ), we impose on  $D(w_N)$  some other set of  $N$  conditions, which ensures that  $D(w_N)$  approaches zero as  $N$  becomes infinite. The collocation method specifies that  $D(w_N)$  be equal to zero at  $N$  points of the region, and the  $N$  equations for the coefficients  $C_i$  are obtained directly without carrying out any integration.

A variation of the collocation method due to Courant (Ref. 24) is to demand that  $\iint_{R_i} D(w_N) dx dy$  be equal to zero over  $N$  subdivided regions  $R_i$  of the given region  $R$ . In other words, we define  $\delta w$  in eq. (6) so that

$$\delta w(x, y) = \begin{cases} 1 & \text{in } R_i, (i = 1, 2, \dots, N), \\ 0 & \text{elsewhere in } R. \end{cases} \quad (8)$$

The  $N$  equations

$$\iint_{R_i} D(w_N) dx dy = 0, (i = 1, 2, \dots, N) \quad (9)$$

are then just enough to determine the constants  $C_i$ .

Finally, in the method of least squares, (Refs. 26, 27 and 28), the constants  $C_i$  are to be determined by requiring that the mean square error be as small as possible. That is,

$$\iint_R [D(w_N)]^2 dx dy = \min. \quad (10)$$

Hence we have  $N$  equations

$$\iint_R D(w_N) \frac{\partial D(w_N)}{\partial C_i} dx dy = 0, (i = 1, 2, \dots, N), \quad (11)$$

to determine the constants  $C_i$ .

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All the methods outlined above are characterized by the fact that the potential energy  $V$  is approached from above by the approximating functions  $w_N = \sum_{i=1}^N C_i \varphi_i$ . In other words, if  $V(w_N)$  is the value of  $V$  when the true solution  $w$  is replaced by  $w_N$ , then

$$V(w_N) \geq V(w). \quad (12)$$

To show this,\* let us define an error function  $e_N(x, y)$  by the equation

$$e_N(x, y) = w(x, y) - w_N(x, y). \quad (13)$$

Evidently  $e_N$  satisfies all the continuity conditions of  $w$  and  $w_N$ . Now

$$\begin{aligned} V(w) = & \frac{D}{2} \iint_R \left\{ (\nabla^2 w)^2 - 2(1-\sigma) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \\ & - \iint_R q w dx dy + \int_L M_n \frac{\partial w}{\partial n} ds - \int_L \left( Q_n - \frac{\partial M_{nt}}{\partial s} \right) w ds \end{aligned} \quad (1)$$

where  $w$  satisfies the field equation

$$\nabla^4 w = \frac{q}{D} \quad \text{in } R \quad (14)$$

and the boundary conditions

$$\begin{aligned} \int_L \left\{ D \left[ \cos^2 \theta \left( \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} \right) + \sin^2 \theta \left( \frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2} \right) + (1-\sigma) \sin 2\theta \frac{\partial^2 w}{\partial x \partial y} \right] + M_n \right\} \frac{\partial w}{\partial n} ds = 0 \\ \int_L \left\{ D \left[ (1-\sigma) \frac{\partial}{\partial s} \left\{ \frac{1}{2} \sin 2\theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right\} \right. \right. \\ \left. \left. - \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \cos \theta - \left( \frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \sin \theta \right] - \left( Q_n - \frac{\partial M_{nt}}{\partial s} \right) \right\} w ds = 0 \end{aligned} \quad (15)$$

Now in all the above methods the approximating functional sequence  $w_N$  is so chosen that eqs. (15) are satisfied when  $w$  is replaced by  $w_N$ .

Writing  $w_N = w - e_N$ , and substituting for  $w$  into eq. (1), we obtain

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\* The particular case of a plate with built-in edges was proved by W-Ritz, (Ref. 18).



$$\begin{aligned}
V(w_N) &= \frac{D}{2} \iint_R \left\{ [\nabla^2(w-e_N)]^2 - 2(1-\sigma) \left[ \frac{\partial^2(w-e_N)}{\partial x^2} \frac{\partial^2(w-e_N)}{\partial y^2} - \left( \frac{\partial^2(w-e_N)}{\partial x \partial y} \right)^2 \right] \right\} dx dy \\
&\quad - \iint_R q(w-e_N) dx dy + \int_L \frac{\partial(w-e_N)}{\partial n} M_n ds - \int_L \left( Q_n - \frac{\partial M_{nt}}{\partial t} \right) (w-e_N) ds \\
&= V(w) + \pi(e_N) - D \iint_R \left\{ \nabla^2 w \nabla^2 e_N - (1-\sigma) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 e_N}{\partial y^2} + \frac{\partial^2 e_N}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right. \right. \\
&\quad \left. \left. - 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 e_N}{\partial x \partial y} \right] - \frac{q}{D} e_N \right\} dx dy \\
&\quad - \int_L M_n \frac{\partial e_N}{\partial n} ds + \int_L \left( Q_n - \frac{\partial M_{nt}}{\partial s} \right) e_N ds.
\end{aligned} \tag{16}$$

where

$$\pi(e_N) = \frac{D}{2} \iint_R \left\{ (\nabla^2 e_N)^2 - 2(1-\sigma) \left[ \frac{\partial^2 e_N}{\partial x^2} \frac{\partial^2 e_N}{\partial y^2} - \left( \frac{\partial^2 e_N}{\partial x \partial y} \right)^2 \right] \right\} dx dy. \tag{17}$$

The first term under the integral sign on the right hand side of eq. (16) can be transformed as follows: Let us recall the Green's formula

$$\iint_R (v \nabla^2 u - u \nabla^2 v) dx dy = \int_L v \frac{\partial u}{\partial n} ds - \int_L u \frac{\partial v}{\partial n} ds$$

where  $u, v$  are any two functions continuous  $(D, 1)$  in  $R$  and on  $L$ . If we change  $v$  into  $\nabla^2 w$ , and  $u$  into  $e_N$ , we obtain:

$$\iint_R \nabla^2 w \nabla^2 e_N dx dy = \iint_R e_N \nabla^4 w dx dy + \int_L \nabla^2 w \frac{\partial e_N}{\partial n} ds - \int_L e_N \frac{\partial \nabla^2 w}{\partial n} ds. \tag{18}$$

Other terms in (16) are transformed through integration by parts as follows:

$$\begin{aligned}
\iint_R \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 e_N}{\partial y^2} dx dy &= - \int_L \frac{\partial^2 w}{\partial x^2} \frac{\partial e_N}{\partial y} dx - \int_L \frac{\partial^2 w}{\partial x \partial y} \frac{\partial e_N}{\partial y} dy + \iint_R \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 e_N}{\partial x \partial y} dx dy, \\
\iint_R \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 e_N}{\partial x^2} dx dy &= + \int_L \frac{\partial^2 w}{\partial y^2} \frac{\partial e_N}{\partial x} dy + \int_L \frac{\partial^2 w}{\partial x \partial y} \frac{\partial e_N}{\partial x} dx + \iint_R \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 e_N}{\partial x \partial y} dx dy.
\end{aligned} \tag{19}$$

Substituting (18) and (19) into (16), we obtain

$$\begin{aligned}
 V(w_N) = & V(w) + U(e_N) - D \iint_R \left( \nabla^4 w - \frac{q}{D} \right) e_N dx dy \\
 & + D(1-\sigma) \left\{ - \int_L \frac{\partial^2 w}{\partial x^2} \frac{\partial e_N}{\partial y} dx - \int_L \frac{\partial^2 w}{\partial x \partial y} \frac{\partial e_N}{\partial y} dy + \right. \\
 & \left. + \int_L \frac{\partial^2 w}{\partial y^2} \frac{\partial e_N}{\partial x} dy + \int_L \frac{\partial^2 w}{\partial x \partial y} \frac{\partial e_N}{\partial x} dx \right\} \\
 & - D \int_L \nabla^2 w \frac{\partial e_N}{\partial n} ds + D \int_L \frac{\partial \nabla^2 w}{\partial n} e_N ds - \int_L M_n \frac{\partial e_N}{\partial n} ds \\
 & + \int_L \left( Q_n - \frac{\partial M_{nt}}{\partial s} \right) e_N ds.
 \end{aligned} \tag{20}$$

The third term on the right hand side of (20) vanishes on account of the field eq. (14). The line integrals can be simplified as follows. Let the angle between the outer normal to L and the x-axis be  $\theta$ . Then along L,

$$\begin{aligned}
 dx &= -\sin \theta ds, & dy &= \cos \theta ds, \\
 \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial n} - \sin \theta \frac{\partial}{\partial s}, & \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial n} + \cos \theta \frac{\partial}{\partial s}, \\
 \frac{\partial}{\partial n} &= \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, & \frac{\partial}{\partial s} &= -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}.
 \end{aligned}$$

Expressing  $\frac{\partial e_N}{\partial x}$ ,  $\frac{\partial e_N}{\partial y}$  in terms of  $\frac{\partial e_N}{\partial n}$ ,  $\frac{\partial e_N}{\partial s}$ , and dx, dy in terms of ds, and using the first eq. of (15), we see that (20) becomes

$$\begin{aligned}
 V(w_N) = & V(w) + U(e_N) \\
 & + D(1-\sigma) \int_L \left[ \sin \theta \cos \theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right] \frac{\partial e_N}{\partial s} ds \\
 & + D \int_L \frac{\partial \nabla^2 w}{\partial n} e_N ds + \int_L \left( Q_n - \frac{\partial M_{nt}}{\partial s} \right) e_N ds.
 \end{aligned} \tag{21}$$

But

$$\begin{aligned}
& \int_L \left[ \sin \theta \cos \theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right] \frac{\partial e_N}{\partial s} ds \\
&= \int_L \frac{\partial}{\partial s} \left\{ \left[ \sin \theta \cos \theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right] e_N \right\} ds \\
&= \int_L \frac{\partial}{\partial s} \left\{ \sin \theta \cos \theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right\} e_N ds.
\end{aligned} \tag{22}$$

The first integral on the right-hand side of (22) vanishes since the integrand is a unique continuous function and since  $L$  is a simple closed curve. The second line integral on the right-hand side of (22) cancels out with the last two terms  $D \int_L \frac{\partial \nabla^2 w}{\partial n} e_N ds + \int_L (Q_n - \frac{\partial M_{nt}}{\partial s}) e_N ds$  of (21) on account of the second of the eq. (15) by observing that

$$Q_n = -D \frac{\partial \nabla^2 w}{\partial n} = -D \left[ \nabla^2 \frac{\partial w}{\partial x} \cos \theta + \nabla^2 \frac{\partial w}{\partial y} \sin \theta \right].$$

Hence we obtain finally

$$V(w_N) = V(w) + \mathcal{U}(e_N). \tag{23}$$

But  $\mathcal{U}(e_N)$  is the strain energy stored in a plate when it is subjected to a fictitious deflection  $w = e_N(x, y)$  and therefore is a positive function. Hence the proof is completed.



### 2.3 OTHER DIRECT METHODS (COURANT-TREFFTZ PROCEDURES).

A counterpart to Rayleigh-Ritz procedure is suggested by Courant (Ref. 15) and developed by Trefftz (Ref. 29) in connection with the Dirichlet's problem.

The essence of their method is to loosen the boundary conditions. Unlike Rayleigh-Ritz method, the approximating functions  $w_N$  are so chosen that they satisfy the differential equation of  $w$  but are subjected to less restrictive supplementary conditions. For concrete illustration consider the plate problem in second order approximation. Let us recall (§2.1) that the vanishing of the first variation of the energy integral  $V$  implies the vanishing of the surface integral

$$\iint_R (\nabla^4 w - q) \delta w \, dx \, dy = 0, \quad (1)$$

and the line integral

$$\int_L B_1(w) \frac{\delta w}{\delta n} \, ds + \int_L B_2(w) \delta w \, ds = 0 \quad (2)$$

where

$$\begin{aligned} B_1(w) &= D \left[ (1-\sigma) \left( \frac{\partial^2 w}{\partial x^2} \cos^2 \theta + \sin 2\theta \frac{\partial^2 w}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 w}{\partial y^2} \right) + \sigma \nabla^2 w \right] + M_n, \\ B_2(w) &= D \left[ (1-\sigma) \frac{\partial}{\partial s} \left\{ \sin \theta \cos \theta \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y} \right\} \right. \\ &\quad \left. - \cos \theta \nabla^2 \frac{\partial w}{\partial x} - \sin \theta \nabla^2 \frac{\partial w}{\partial y} \right] - (Q_n - \frac{\partial M_{nt}}{\partial s}). \end{aligned} \quad (3)$$

$\delta w$  and  $\frac{\delta w}{\delta n}$  are arbitrary functions except on restrained (rigid) edges when they are required to take preassigned values as well as  $w$  and  $\partial w / \partial n$ . Now let us take a sequence of approximating functions (cf. end of §2.1)

$$w_N = \sum_{i=1}^N b_i g_i(x, y), \quad (N=1, 2, 3, \dots) \quad (4)$$

where  $b_i$  are unknown constants and  $g_i(x, y)$ , the so-called coordinate functions of  $w_N$ , are functions so chosen that  $w_N$  satisfies the differential equation  $\nabla^4 w - q/p = 0$ , and hence the eq. (1). Therefore, we are left only with the line integral (2) to worry about. Now eq. (2) could not be satisfied unless  $w_N$  is a

true solution. But we can minimize the error induced by replacing  $w$  by  $w_N$  by a proper choice of the constants  $b_i$ . Now if  $N$  tends to infinity and  $w_N$  does converge to a function  $w$ , and if  $g_i(x, y)$  form a complete system of functions, then the solution is established.

Courant and Trefftz's original idea is to devise a method in which the minimizing integral ( $V(w)$  of §2.1) is to be approached from below, i.e.,  $V(w_N) \leq V(w)$  for any  $N$ , so that this method is a true counterpart to Rayleigh-Ritz procedure, in which the minimizing integral is approached from above. (cf. §2.2). For plate problems this object can be achieved in certain cases by a special formulation of the procedure to determine  $b_i$ . But in general this is not the most convenient procedure. As our main aim is to obtain useful approximate solutions, let us be not confined to any definite manner of approaching the minimizing integral. Then we can formulate the detail procedure in many ways. Each method in the preceding section has a counterpart in the present one. In practical applications the most advantageous way is usually secured by a combination of several methods.

It will be convenient for further discussion to distinguish the natural boundary conditions from the rigid boundary conditions. The latter refers to rigid or supported edges of a plate. The nomenclature is due to Courant (Ref. 14). On rigid boundaries the support condition puts on a first-hand restriction on the values of  $w$  or its first derivatives. Thus,  $w = \frac{\partial w}{\partial n} = 0$  on a built-in edge. Now in our former variational equations (1) and (2)  $\delta w$  and  $\frac{\partial \delta w}{\partial n}$  are not arbitrary on restrained edges, but are required to satisfy the restraining conditions on displacements. For example,  $\delta w$  and  $\frac{\partial \delta w}{\partial n}$  must vanish on a built-in edge. To take off this restriction on the otherwise perfectly arbitrary variation  $\delta w$ , let us write the boundary condition for such an edge in the form

$$\int_{L_i} \frac{\partial w}{\partial n} \delta_1 w ds + \int_{L_i} w \delta_2 w ds = 0$$

where  $L_i$  is a built-in edge. Then  $\delta_i w$ , ( $i=1, 2$ ) are entirely unrestricted. A built-in edge is rigid with respect to slope and deflection, a simply-supported edge is rigid with respect to deflection alone. Let us denote the part of  $L$  which is natural with respect to slope by  $L_{ns}$ , that natural with respect to deflection by  $L_{nd}$ , and the part which is rigid with respect to slope by  $L_{rs}$ , and that rigid with respect to deflection by  $L_{rd}$ . Equation (2) can now be conveniently written as

$$\int_{L_{ns}} B_1(w) \delta_1 w ds + \int_{L_{nd}} B_2(w) \delta_2 w ds + \int_{L_{rs}} \frac{\partial w}{\partial n} \delta_3 w ds + \int_{L_{rd}} w \delta_4 w ds = 0 \quad (5)$$

where  $\delta_i w$ , ( $i=1, 2, 3, 4$ ) are now perfectly arbitrary small quantities.

Analogy to Galerkin's method can now be formulated as follows: Let the coordinate functions  $g_i(x, y)$  be so chosen that the approximate solution  $w_N = \sum_{i=1}^N b_i g_i(x, y)$  satisfies the differential

eq.  $\nabla^4 w - \frac{q}{D} = 0$ . But eq. (5) cannot be satisfied by  $w_N$  if  $w_N$  is not a true solution. Now let us loosen the boundary conditions in such a way that although (5) is not true for an infinite variety of  $\delta w$ , yet it is satisfied for  $N$  variations  $k g_i(x, y)$ , ( $i=1, 2, \dots, N$ ) of  $\delta w$ ,  $k$  being an arbitrary constant. Then when  $N$  tends to infinity and if  $g_i(x, y)$  form a set of complete system of functions, the system of all equations

$$\int_{Lns} B_1(w_N) g_i ds + \int_{Lnd} B_2(w_N) g_i ds + \int_{Lrs} \frac{\partial w_N}{\partial n} g_i ds + \int_{Lrd} w_N g_i ds = 0 \quad (6)$$

$$(i = 1, 2, \dots, N)$$

becomes equivalent to the relation (5).

Eqs. (6) are  $N$  in number and linear in  $b_i$ , from which  $b_i$  and hence  $w_N$  can be determined.

-----

The collocation method is obviously valid. Consider  $\delta_i w$ , ( $i=1, 2, 3, 4$ ) in eq. (5) to be zero everywhere except at  $N$  discrete points along the boundary  $L$ . But at these  $N$  points let  $\delta_i w$  take arbitrary constants. Then in order that (5) can be satisfied in this particular case we must have the coefficients of  $\delta_i w$ , ( $i=1, 2, 3, 4$ ) in the integrands of eq. (5) to vanish at these  $N$  points. From this condition we obtain  $N$  equations for the determination of the  $N$  constants without any integration. In fact we are satisfying the boundary condition at  $N$  points on  $L$ .

-----

A modification to the collocation method is effected by dividing  $L$  into  $N$  parts, and consider  $\delta w$  to be a constant in each part  $L_k$  but zero everywhere else. Then from (5) we have  $N$  equations

$$\sum_{j=1}^4 \int_{L_k} B_j(w_N) ds = 0, \quad (k=1, 2, \dots, N) \quad (7)$$

to determine  $b_i$ , ( $i=1, 2, \dots, N$ ). In this case we satisfy the boundary conditions by a piecewise averaging process.

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The method of least squares can be applied directly. Consider  $B_i(w_N)$  as error functions, we try to determine the constants  $b_i$  by minimizing the error

$$e(w_N) = \int_{Lns} B_1^2(w_N) ds + \int_{Lnd} B_2^2(w_N) ds + \int_{Lrs} \left(\frac{\partial w_N}{\partial n}\right)^2 ds + \int_{Lrd} w_N^2 ds \quad (8)$$

By usual process, we obtain N equations:

$$0 = \frac{1}{2} \frac{\partial e(w_N)}{\partial b_i} = \int_{Lns} B_1(w_N) B_1(g_i) ds + \int_{Lnd} B_2(w_N) B_2(g_i) ds + \int_{Lrs} \frac{\partial w_N}{\partial n} g_i ds + \int_{Lrd} w_N g_i ds, \quad (i=1, 2, \dots, N), \quad (9)$$

since  $\frac{\partial w_N}{\partial b_i} = g_i(x, y)$ .

Evidently  $e(w_N)$  vanishes for the true solution  $w$  of the problem.

-----

Now let us consider under what conditions are the above mentioned methods a counterpart to the theorem proved at the end of §2.2, that is, the minimizing integral is approached from below. In other words,

$$V(w_N) \leq V(w). \quad (10)$$

Let us again define an error function  $e_N$  by

$$e_N = w(x, y) - w_N(x, y). \quad (11)$$

Then

$$\begin{aligned} V(w) &= V(w_N + e_N) \\ &= V(w_N) + \pi(e_N) \\ &\quad + D \iint_R \left\{ \nabla^2 w_N \nabla^2 e_N + (1-\sigma) \left[ \frac{\partial^2 w_N}{\partial x^2} \frac{\partial^2 e_N}{\partial y^2} + \frac{\partial^2 e_N}{\partial x^2} \frac{\partial^2 w_N}{\partial y^2} - 2 \frac{\partial^2 w_N}{\partial x \partial y} \frac{\partial^2 e_N}{\partial x \partial y} \right] - \frac{q}{D} e_N \right\} dx dy \\ &\quad - \int_L M_n \frac{\partial e_N}{\partial n} ds + \int_L \left( Q_n - \frac{\partial M_n}{\partial s} \right) e_N ds. \end{aligned} \quad (12)$$

where  $V(w)$  is given by eq. (2.1:1) and  $\pi(e_N)$  is given by (2.1:17).

Transform the integrals on the right hand side of (12) in the same way as in 2.1, and remembering that

$$\nabla^4 w_N - \frac{q}{D} = 0 \quad \text{in } R,$$

we obtain:

$$V(w) = V(w_N) + U(e_N) + \int_L B_1(w_N) \frac{\partial e_N}{\partial n} ds + \int_L B_2(w_N) e_N ds \quad (13)$$

The last two line integrals are minimized in the process of determining the unknown constants  $b_i$  of  $w_N$ . But in general they do not vanish. In case they do, then eq. (10) holds.

Trefftz suggested (Ref. 30) to use the mean values of  $V$  calculated by Rayleigh-Ritz method and his own as a better approximation to the true minimum of  $V$ . This refinement is important when the value of  $V$  itself is important, as in calculating the rigidity of a shaft in torsion. Another method of estimating the error of approximation is suggested by Friedrichs (Ref. 31) who forms an auxiliary integral the maximum of which is less or equal than the minimum of  $V$ . Its application to torsion problem will be discussed in Chapter 4.

### III

#### THEORY OF SWEEPED FLAT PLATES

##### 3.1 Introduction

In second order (first) approximation the problem of bending of flat plates consists in integrating the equation

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} q(x, y) \quad (1)$$

where  $w(x, y)$  satisfies along the boundary  $L$  of a region  $R$  certain boundary conditions defined by the problem, and  $w$  is continuous together with its partial derivatives up to fourth order in  $R$  and on  $L$ , (we describe this fact by saying that  $w$  is continuous  $(D, 4)$  in  $R$  and on  $L$ ), and  $q$  satisfies desirable continuity assumptions.

There are four kinds of problems, classified according to the nature of prescribed boundary conditions:

(A). Built-in edge problem,  $w = \frac{\partial w}{\partial n} = 0$  on  $L$ , (2)

where  $n$  is the outer normal to  $L$ .

(B). Supported edge problem,

$$w = 0, \quad \frac{\partial^2 w}{\partial n^2} + \sigma \frac{\partial^2 w}{\partial t^2} = f(x, y) \text{ on } L, \quad (3)$$

where  $n$  and  $t$  are outer normal and tangent to  $L$  respectively, the sense of  $n$  is to  $t$  as  $x$  is to  $y$  axis.

Subcase  $(B_1)$ . Simply-supported edge problem,

— particular case of (3) when  $f = 0$

(C). Unsupported edge problem,

$$\begin{cases} \frac{\partial^2 w}{\partial n^2} + \sigma \frac{\partial^2 w}{\partial t^2} = g(x, y), \\ \frac{\partial^3 w}{\partial n^3} + (2 - \sigma) \frac{\partial^3 w}{\partial n \partial t^2} = h(x, y) \end{cases} \text{ on } L. \quad (4)$$

Subcase  $(C_1)$ . Free-edge problem,

— the particular case of (4) when  $g(x, y) = h(x, y) \equiv 0$

(D). Mixed edge-condition problems.

This can be further classified into

- (D<sub>1</sub>) BS. When a part of  $L$  is simply-supported and other part built-in.
- (D<sub>2</sub>) BU. When  $L$  is partly built in and partly unsupported
- (D<sub>3</sub>) SU. When  $L$  is partly supported and partly unsupported
- (D<sub>4</sub>) SBU. When  $L$  is partly built in, partly supported, and partly unsupported.



Problem (A) is often made a subject of mathematical investigation. Existence theorems and solutions in fairly general forms have been obtained by Almansi, Lauricella, J. Hadamard, A. Dorn, and others. (Refs. 32, 33, 34, 35) Approximate solutions for rectangular plates have been given by W. Ritz and others (Ref. 17). For other solutions reference should be made to the book of Timoshenko (Ref. 10).

Problem (B) is widely studied by Timoshenko and others. (Ref. 10) For rectangular plates an application of Fourier series makes the problem particularly simple. Both problems (A) and (B) can be reduced to a set of first boundary value problems of the potential theory (Ref. 36).

Problem (C) is much more difficult than the first two. Few existing results can be quoted. The problem of vibration of a rectangular plate with free edges which is akin to  $(C_1)$  was studied by Ritz in his 1909 paper in which he illustrated a spectacular success of his energy method.

Of problems (D), a few cases of  $(D_1)$  and  $(D_3)$  for rectangular plates have been studied. (Ref. 10). For other problems, there is little available results.

Our sweep-back wing problem is a problem  $(D_2)$  with BU boundary. Exact solutions for this problem are in general hard to get. But it is possible to find approximate solutions of engineering significance in the form  $w_N = \sum b_i g_i(x, y)$  with the help of some minimal properties of certain related integrals. (cf. Chapter II).

### 3.2 Solution of the differential Equation

It is usually convenient to write the solution of eq. (3.1.1) in the form of a sum of a particular integral and a biharmonic function:

$$w(x, y) = w_p(x, y) + w_c(x, y) \quad (1)$$

so that

$$\frac{\partial^4 w_p}{\partial x^4} + 2 \frac{\partial^4 w_p}{\partial x^2 \partial y^2} + \frac{\partial^4 w_p}{\partial y^4} = \frac{1}{D} q(x, y),$$

and

$$\frac{\partial^4 w_c}{\partial x^4} + 2 \frac{\partial^4 w_c}{\partial x^2 \partial y^2} + \frac{\partial^4 w_c}{\partial y^4} = 0 \quad (2)$$

The particular integral  $w_p$  can be easily obtained if  $q(x, y)$  is simple, say, being a constant, a function of single variable  $x$  or  $y$ , or a polynomial in  $x, y$ . For other cases it was shown by Mathieu, (Ref. 37), that a general expression for  $w_p$  can be obtained by an extension of the potential theory. Let us consider a two-dimensional potential given by the double integral

$$V = \iint_R \log \frac{1}{r} q(a, b) da db \quad (3)$$

extended over a region  $R$ , where  $r$  denotes the distance between the points  $(x, y)$  and  $(a, b)$ , i.e.

$$r^2 = (x-a)^2 + (y-b)^2, \quad (4)$$

and  $q$  satisfies certain integrability conditions. It is well-known that (see, for example Ref. 35)

$$\nabla^2 V = 0 \quad \text{or} \quad -2\pi q(x, y), \quad (\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \quad (5)$$

according to whether the point  $(x, y)$  is situated outside or inside of the region  $R$  respectively.

Observe that

$$\begin{aligned} \nabla^2 \left( r^2 \log \frac{1}{r} + \frac{r^2}{2} \right) &= 4 \log \frac{1}{r} \\ \nabla^4 \left( r^2 \log \frac{1}{r} + \frac{r^2}{2} \right) &= 0 \end{aligned} \quad (6)$$

and consider the function

$$W = \iint_R \left( r^2 \log \frac{1}{r} + \frac{r^2}{2} \right) q(a, b) da db \quad (7)$$

We see that

$$\nabla^2 W = 4V \quad (8)$$

and therefore

$$\nabla^4 w(x, y) = \begin{cases} 0 & \text{if the point } (x, y) \text{ is outside of } R, \\ -8\pi q(x, y) & \text{if } (x, y) \text{ is inside of } R. \end{cases}$$

Therefore, we obtain the general result:

$$w_p(x, y) = -\frac{1}{8\pi D} \iint_R \left( r^2 \log \frac{1}{r} + \frac{r^2}{2} \right) q(a, b) da db \quad (9)$$

where

$$r^2 = (x-a)^2 + (y-b)^2.$$

For the biharmonic function  $w_c$ , we have the following general theorems, (Ref. 36, Chapter 19).

If  $w_1$  and  $w_2$  are two harmonic functions in a region  $R$ , then

$$w = x w_1 + w_2 \quad (10)$$

is a biharmonic function in the same region.

Conversely, if the boundary  $L$  of  $R$  intersects every line parallel to  $x$  axis in at most two points, then for every biharmonic function  $w$  in  $R$ , there exist two harmonic functions  $w_1$  and  $w_2$ , so that  $w$  can be represented by the formula (10).

Similarly, if every line parallel to  $y$ -axis intersects the boundary  $L$  of  $R$  in at most two points, we can represent every biharmonic function in  $R$  by the form

$$w = y w_1 + w_2 \quad (11)$$

where  $w_1$  and  $w_2$  are harmonic functions in  $R$ . Conversely, every function  $w$  in this form (11) is biharmonic in  $R$ .

Furthermore, if the origin is enclosed in  $R$  and every radius vector intersects  $L$  in only one point, then every biharmonic function  $w$  in  $R$  can be represented by functions of the form

$$w = (r^2 - r_0^2) w_1 + w_2, \quad (12)$$

where

$$r^2 = x^2 + y^2,$$

$w_1$  and  $w_2$  are two harmonic functions in  $R$ , and  $r_0$  is any preassigned constant. Conversely, every function  $w$  in this form is a biharmonic function in  $R$ .



These theorems enable us to express a biharmonic function by harmonic functions, for which we have a very complete store of information.

A consequence of these theorems is that every biharmonic function is analytic, since every harmonic function is.

Our swept plate problem will be attacked in the following manner: In each case a suitable form of particular solution  $w_p$  will first be obtained. This  $w_p$ , while balancing the distributed normal loads over the faces of the plate, will induce bending moments and shear reactions along the edges of the plate. The next step is then to find a solution for a plate loaded along the edge: not only loaded by the actual external load but also by the fictitious load due to  $w_p$ . Now since by Mathieu's theorem a  $w_p$  can always be found if the normal loading observes certain integrability conditions, it is readily seen that the second problem, a plate loaded along the edges, is of a very fundamental nature. According to this observation we shall attack the problem of swept plates loaded at tip section first, then of that loaded at the leading and trailing edges, and finally general distributed loads.

### 3.3 Position of Critical Stresses

When the function  $q(x, y)$  is null over a subregion  $R_i$  of  $R$ , the differential equation of deformation (eq. 3.1.1) becomes

$$\nabla^2(\nabla^2 w) = 0 \quad \text{in } R_i. \quad (1)$$

Hence  $\nabla^2 w$  is a harmonic function over  $R_i$ .

But a harmonic function in  $R_i$  cannot have a maximum or minimum at a point in the interior of the region  $R_i$ . (For proof, see, for example, §37, Ref. 38). Its extremum is reached only on the boundary of  $R_i$ . Singular points in  $R$  can be regarded as boundary points. The points under concentrated loading are boundary points in this sense.

Now  $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$  is the mean curvature of the bent middle surface of the plate. Hence, the above statement means that the mean curvature of a bent plate under normal loading reaches its maximum or minimum either on the edge of the plate or under the points of loading.

When external load is applied on to a plate only along the edges of the plate, the maximum mean curvature is assured to reach its maximum only on the edges. It was remarked at the end of the last section that all swept plate problems can be reduced to problems of this nature.

The maximum mean curvature is related to the maximum bending stresses along the edges in the following way: If an edge is built-in, or is supported, then  $w = 0$  on that edge. So  $\nabla^2 w = \frac{\partial^2 w}{\partial n^2} = M_n / D$ , where  $M_n$  is the bending moment per unit length acting on the edge. Hence on such an edge, the maximum in mean curvature means the maximum in bending moment and so also the maximum bending stresses.

Moreover, since  $M_x = D \left( \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} \right)$ ,  $M_y = D \left( \frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2} \right)$  we have

$$\nabla^2 w = \frac{1}{D(1+\sigma)} (M_x + M_y). \quad (2)$$

Now on a free edge, the bending moment  $M_n$  is zero. Hence

$$\nabla^2 w = \frac{1}{D(1+\sigma)} M_s \quad \text{on a free edge.} \quad (3)$$

$M_s$  is the bending moment in the plate. Therefore again the maximum mean curvature is proportional to the maximum bending stresses.

To sum up, these facts indicate that the critical point for stresses is either under the load or on the boundary of the plate.

Similar reasoning leads to corresponding theorems regarding to shearing stress. If, in  $R_i$ ,  $q$  is a function of  $y$  alone, then

$$\frac{\partial}{\partial x} \nabla^2 (\nabla^2 w) = \nabla^2 \left( \nabla^2 \frac{\partial w}{\partial x} \right) = \frac{1}{D} \frac{\partial q(y)}{\partial x} = 0 \quad (4)$$

But  $-D \nabla^2 \frac{\partial w}{\partial x} = Q_x$ , the vertical shearing force on  $x = \text{const.}$  sections.

Hence  $\nabla^2 Q_x = 0$  in  $R_i$  if  $q = q(y)$ . (5)

In the same way, if  $q$  is a function of  $x$  alone, then

$$\nabla^2 Q_y = 0 \quad \text{in } R_i, \quad (q = q(x).) \quad (6)$$

$Q_x$  and  $Q_y$  are then harmonic functions in these cases. They reach their maximum or minimum only on the boundary of  $R_i$ . In other words,  $Q_x$  (or  $Q_y$ ) reaches its extremum either on the edge of the plate or at points where  $\frac{\partial q}{\partial x}$  (or  $\frac{\partial q}{\partial y}$ ) is different from zero.

The significance of these theorems will be appreciated in the application of Courant-Trefftz methods. They tell us where the stresses are likely to be critical. In fact, in the way Trefftz method is applied below, they imply that the accuracy and convergence of the process need only be checked on the boundary of the plate. For, let us define the error function in the same way as was done in § 2.3.

$$e_N = w - w_N$$

Where  $w_N$  is an approximating function to the true solution  $w$ , then since in Courant-Trefftz method both  $w$  and  $w_N$  satisfy the differential equation  $\nabla^4 w = q/D$ , we have

$$\nabla^4 e_N = 0 \quad \text{in } R.$$

Therefore the mean curvature of the error function reaches the extremum only on the boundary. From what has been said above we see that the maximum error in stresses occurs only on the boundary of the plate.